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and non-conforming FE schemes) fit into the HDM. Finally, some technical results are gathered in an appendix.

Notations. A fourth order symmetric *tensor* P is a linear map $\mathcal{S}_d(\mathbb{R}) \to \mathcal{S}_d(\mathbb{R})$, where $\mathcal{S}_d(\mathbb{R})$ is the set of symmetric matrices, d is the dimension; p_{ijkl} denote the indices of the fourth order tensor P in the canonical basis of $\mathcal{S}_d(\mathbb{R})$. For simplicity, we follow the Einstein summation convention unless otherwise stated, i.e, if an index is repeated in a product, summation is implied over the repeated index. For $\xi \in \mathcal{S}_d(\mathbb{R})$, using the definition of symmetric tensor, one has $P\xi \in \mathcal{S}_d(\mathbb{R})$ and $p_{ijkl} = p_{jikl} = p_{ijlk}$. The scalar product on $\mathcal{S}_d(\mathbb{R})$ is defined by $\xi : \phi = \xi_{ij}\phi_{ij}$. For a function $\xi : \Omega \to \mathcal{S}_d(\mathbb{R})$, denoting the Hessian matrix by $\mathcal{H}$ we set $\mathcal{H} : \xi = \partial_{ij}\xi_{ij}$. Finally, the transpose P^τ of P is given by $P^\tau = (p_{klij})$, if $P = (p_{ijkl})$. Note that $P^\tau\xi : \phi = \xi : P\phi$. The tensor product $a \otimes b$ of two vectors $a, b \in \mathbb{R}^d$ is the 2-tensor with coefficients $a_i b_j$. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$, as is the induced norm on $\mathcal{S}_d(\mathbb{R})$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^d$ is denoted by $|E|$ (note that the nature of the argument of $|\cdot|$, a vector or a set, makes it clear if we talk about the Euclidean norm or the Lebesgue measure). The norm in $L^2(\Omega)$, $L^2(\Omega)^d$ for vector-valued functions, and $L^2(\Omega; \mathbb{R}^{d\times d})$ for matrix-valued functions, is denoted by $\|\cdot\|$.

2. Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega$ and consider the following fourth order model problem with clamped boundary conditions.

$$\sum_{i,j,k,l=1}^{d} \partial_{kl}(a_{ijkl}\partial_{ij}\overline{u}) = f \quad \text{in } \Omega, \tag{2.1a}$$

$$\overline{u} = \frac{\partial \overline{u}}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2.1b}$$

where $\boldsymbol{x} = (x_1, x_2, ..., x_d) \in \Omega$, $f \in L^2(\Omega)$, n is the unit outer normal to Ω and the coefficients a_{ijkl} are measurable bounded functions which satisfy the conditions $a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}$ for $i, j, k, l = 1, \cdots, d$. For all $\xi, \phi \in \mathcal{S}_d(\mathbb{R})$, we assume the existence of a fourth order tensor B such that $A\xi : \phi = B\xi : B\phi$, where A is the four-tensor with indices a_{ijkl}. We notice that $B\xi : B\phi = B^\tau B\xi : \phi$, so that $A = B^\tau B$.

Setting

$$\begin{aligned} V = H_0^2(\Omega) &= \left\{ v \in H^2(\Omega); v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \right\} \\ &= \left\{ v \in H^2(\Omega); v = |\nabla v| = 0 \text{ on } \partial\Omega \right\}, \end{aligned}$$

the weak formulation of (2.1) is

$$\text{Find } \overline{u} \in V \text{ such that } \forall v \in V, \quad \int_\Omega \mathcal{H}^B\overline{u} : \mathcal{H}^B v \, \mathrm{d}\boldsymbol{x} = \int_\Omega f v \, \mathrm{d}\boldsymbol{x}, \tag{2.2}$$

where $\mathcal{H}^B v = B\mathcal{H}v$. Note that $\int_\Omega \mathcal{H}^B\overline{u} : \mathcal{H}^B v \, \mathrm{d}\boldsymbol{x} = \int_\Omega A\mathcal{H}\overline{u} : \mathcal{H}v \, \mathrm{d}\boldsymbol{x}$, since $A = B^\tau B$. We assume in the following that B is constant over Ω, and that the following coercivity property holds:

$$\exists \varrho > 0 \text{ such that } \|\mathcal{H}^B v\| \geq \varrho \|v\|_{H^2(\Omega)}, \ \forall v \in H_0^2(\Omega). \tag{2.3}$$

Hence, the weak formulation (2.2) has a unique solution by the Lax–Milgram lemma.

Remark 2.1. *Adapting the analysis of Section 3 to B dependent on $\boldsymbol{x} \in \Omega$ is easy, provided that the entries of B belong to $W^{2,\infty}(\Omega)$.*

2.1. **Examples.** Let us examine two specific examples of the abstract problem (2.1).

2.1.1. *Biharmonic problem.* The biharmonic problem is

$$\Delta^2 u = f \text{ in } \Omega, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega. \tag{2.4}$$

The weak formulation of this model is given by (2.2) provided that B is chosen to satisfy

$$\int_\Omega \mathcal{H}^B u : \mathcal{H}^B v \,\mathrm{d}\boldsymbol{x} = \int_\Omega \Delta u \Delta v \,\mathrm{d}\boldsymbol{x}.$$

One possible choice of B is therefore to set $B\xi = \frac{\mathrm{tr}(\xi)}{\sqrt{d}}\mathrm{Id}$ for $\xi \in \mathcal{S}_d(\mathbb{R})$ (where Id is the identity matrix), in which case $\mathcal{H}^B = \Delta$. Since $\int_\Omega \Delta u \Delta v \,\mathrm{d}\boldsymbol{x} = \int_\Omega \mathcal{H}u : \mathcal{H}v \,\mathrm{d}\boldsymbol{x}$, another possibility is to set B the identity tensor ($B\xi = \xi$), in which case $\mathcal{H}^B = \mathcal{H}$. By the Poincaré inequality, both choices satisfy (2.3).

2.1.2. *Plate problem.* The clamped plate problem [6, Chapter 6] corresponds to (2.2) with $d = 2$ and left-hand side

$$\int_\Omega \Delta u \Delta v + (1-\gamma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v) \,\mathrm{d}\boldsymbol{x}. \tag{2.5}$$

Here, the constant γ lies in the interval $(0, \frac{1}{2})$. We notice that (2.5) is equal to $\int_\Omega A\mathcal{H}u : \mathcal{H}v \,\mathrm{d}\boldsymbol{x}$, where the tensor A has non-zero indices $a_{1111} = 1$, $a_{2222} = 1$, $a_{1212} = (1-\gamma)$, $a_{2121} = (1-\gamma)$, $a_{1122} = \gamma$ and $a_{2211} = \gamma$. Its 'square root' can be defined as the tensor B with non-zero indices $b_{1111} = b_{2222} = \sqrt{\frac{1+\sqrt{1-\gamma^2}}{2}}$, $b_{1122} = b_{2211} = \sqrt{\frac{1-\sqrt{1-\gamma^2}}{2}}$ and $b_{1212} = b_{2121} = \sqrt{1-\gamma}$. It can be checked that (2.3) holds since, for some $\varrho > 0$, $A\boldsymbol{\xi} : \boldsymbol{\xi} \geq \varrho^2|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathcal{S}_d(\mathbb{R})$.

3. The Hessian discretisation method

We present here the Hessian discretisation method, and list the properties that are required for the convergence analysis of the Hessian scheme. The error estimate is stated at the end of the section.

Definition 3.1 (B–Hessian discretisation)**.** *A B–Hessian discretisation for clamped boundary conditions is a quadruplet $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D}, \mathcal{H}^B_\mathcal{D})$ such that*

- *$X_{\mathcal{D},0}$ is a finite-dimensional space encoding the unknowns of the method,*
- *$\Pi_\mathcal{D} : X_{\mathcal{D},0} \to L^2(\Omega)$ is a linear mapping that reconstructs a function from the unknowns,*
- *$\nabla_\mathcal{D} : X_{\mathcal{D},0} \to L^2(\Omega)^d$ is a linear mapping that reconstructs a gradient from the unknowns,*
- *$\mathcal{H}^B_\mathcal{D} : X_{\mathcal{D},0} \to L^2(\Omega; \mathbb{R}^{d\times d})$ is a linear mapping that reconstructs a discrete version of $\mathcal{H}^B (= B\mathcal{H})$ from the unknowns. It must be chosen such that $\|\cdot\|_\mathcal{D} := \|\mathcal{H}^B_\mathcal{D} \cdot\|$ is a norm on $X_{\mathcal{D},0}$.*

Remark 3.2 (Dependence of the Hessian discretisation on B)**.** *In the (2nd order) gradient discretisation method, the definition of a gradient discretisation is independent of the differential operator. Here, our definition of Hessian discretisation depends on B, that appears in the differential operator. This is justified by the fact that some methods (such as the one presented in Section 5) are not built on an approximation of the entire Hessian of the functions, but only on some of their derivatives (such as the Laplacian of the functions). Although it might be possible to enrich these methods by adding approximations of the 'missing' second order derivatives (as done in [9] in the context of the GDM), it does not seem to be the most natural way to proceed, and it leads to additional technicality in the analysis. Making the definition of HD dependent on the considered model through B enables us to more naturally embed some known methods into the HDM.*
Note however that a number of FE methods provide approximations of the entire Hessian of the functions (see Sections 4 and 7). For those methods, a B-Hessian discretisation is built from an Id*-Hessian discretisation (that is independent of the model) by setting $\mathcal{H}_{\mathcal{D}}^{B}=B\mathcal{H}_{\mathcal{D}}^{\rm Id}$.*

If $\mathcal{D}=(X_{\mathcal{D},0},\Pi_{\mathcal{D}},\nabla_{\mathcal{D}},\mathcal{H}_{\mathcal{D}}^{B})$ is a B–Hessian discretisation, the corresponding scheme for (2.1), called Hessian scheme (HS), is given by

$$\begin{aligned}&\text{Find } u_{\mathcal{D}}\in X_{\mathcal{D},0} \text{ such that for any } v_{\mathcal{D}}\in X_{\mathcal{D},0},\\ &\int_\Omega \mathcal{H}_{\mathcal{D}}^{B}u_{\mathcal{D}}:\mathcal{H}_{\mathcal{D}}^{B}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}=\int_\Omega f\Pi_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{3.1}$$

This HS is obtained by replacing, in the weak formulation (2.2), the continuous space V by $X_{\mathcal{D},0}$, and by using the reconstructions $\Pi_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{D}}^{B}$ in lieu of the function and its Hessian.
We will show that the accuracy of the HS can be evaluated using only three measures, all intrinsic to the Hessian discretisation. The first one is a constant, $C_{\mathcal{D}}^{B}$, which controls the norm of the linear mappings $\Pi_{\mathcal{D}}$ and $\nabla_{\mathcal{D}}$.

$$C_{\mathcal{D}}^{B}=\max_{w\in X_{\mathcal{D},0}\backslash\{0\}}\left(\frac{\|\Pi_{\mathcal{D}}w\|}{\|\mathcal{H}_{\mathcal{D}}^{B}w\|},\frac{\|\nabla_{\mathcal{D}}w\|}{\|\mathcal{H}_{\mathcal{D}}^{B}w\|}\right).\tag{3.2}$$

The second measure of accuracy is the interpolation error $S_{\mathcal{D}}^{B}$ defined by

$$\begin{aligned}&\forall\,\varphi\in H_0^2(\Omega)\,,\\ &S_{\mathcal{D}}^{B}(\varphi)=\min_{w\in X_{\mathcal{D},0}}\Big(\|\Pi_{\mathcal{D}}w-\varphi\|+\|\nabla_{\mathcal{D}}w-\nabla\varphi\|+\|\mathcal{H}_{\mathcal{D}}^{B}w-\mathcal{H}^{B}\varphi\|\Big).\end{aligned}\tag{3.3}$$

Finally, the third quantity is a measure of limit-conformity of the HD, that is, how well a discrete integration-by-parts formula is verified by the discrete operators:

$$\begin{aligned}&\forall\,\xi\in H^{B}(\Omega):=\{\zeta\in L^2(\Omega)^{d\times d}\,;\,\mathcal{H}:B^\tau B\zeta\in L^2(\Omega)\}\,,\\ &W_{\mathcal{D}}^{B}(\xi)=\max_{w\in X_{\mathcal{D},0}\backslash\{0\}}\frac{1}{\|\mathcal{H}_{\mathcal{D}}^{B}w\|}\left|\int_\Omega\Big((\mathcal{H}:B^\tau B\xi)\Pi_{\mathcal{D}}w-B\xi:\mathcal{H}_{\mathcal{D}}^{B}w\Big)\,\mathrm{d}\boldsymbol{x}\right|.\end{aligned}\tag{3.4}$$

Note that if $\xi\in H^{B}(\Omega)$ and $\phi\in H_0^2(\Omega)$, integration-by-parts show that $\int_\Omega(\mathcal{H}:B^\tau B\xi)\phi=\int_\Omega B\xi:\mathcal{H}^{B}\phi$. Hence, the quantity in the right-hand side of (3.4) measures a defect of discrete integration-by-parts between $\Pi_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{D}}^{B}$.
Closely associated to the three measures above are the notions of coercivity, consistency and limit-conformity of a sequence of Hessian discretisations.

Definition 3.3 (Coercivity, consistency and limit-conformity)**.** *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of B–Hessian discretisations in the sense of Definition 3.1. We say that*

(1) $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *is* coercive *if there exists $C_P \in \mathbb{R}^+$ such that $C^B_{\mathcal{D}_m} \le C_P$ for all $m \in \mathbb{N}$.*

(2) $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *is* consistent, *if*

$$\forall \varphi \in H^2_0(\Omega)\,,\ \lim_{m\to\infty} S^B_{\mathcal{D}_m}(\varphi) = 0. \tag{3.5}$$

(3) $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *is* limit-conforming, *if*

$$\forall \xi \in H^B(\Omega)\,,\ \lim_{m\to\infty} W^B_{\mathcal{D}_m}(\xi) = 0. \tag{3.6}$$

Remark 3.4. *As for the (2nd order) gradient discretisation method, see* [10, Lemmas 2.16 and 2.17]*, it is easily proved that, for coercive sequences of HDs, the consistency and limit-conformity properties* (3.5) *and* (3.6) *only need to be tested for functions in dense subsets of $H^2_0(\Omega)$ and $H^B(\Omega)$, respectively.*

Remark 3.5. *If $B = \mathrm{Id}$, we write $\mathcal{H}_{\mathcal{D}}$ (resp. $C_{\mathcal{D}}$, $S_{\mathcal{D}}$ and $W_{\mathcal{D}}$) instead of $\mathcal{H}^{\mathrm{Id}}_{\mathcal{D}}$ (resp. $C^{\mathrm{Id}}_{\mathcal{D}}$, $S^{\mathrm{Id}}_{\mathcal{D}}$ and $W^{\mathrm{Id}}_{\mathcal{D}}$).*

We can now state our main theorem giving the error estimates.

Theorem 3.6 (Error estimate for Hessian schemes)**.** *Under Assumption* (2.3)*, let $\overline{u}$ be the solution to* (2.2)*. Let $\mathcal{D}$ be a B–Hessian discretisation and $u_{\mathcal{D}}$ be the solution to the corresponding Hessian scheme* (3.1)*. Then we have the following error estimates:*

$$\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\| \le C_{\mathcal{D}} W^B_{\mathcal{D}}(\mathcal{H}\overline{u}) + (C_{\mathcal{D}} + 1) S^B_{\mathcal{D}}(\overline{u}), \tag{3.7}$$

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla\overline{u}\| \le C_{\mathcal{D}} W^B_{\mathcal{D}}(\mathcal{H}\overline{u}) + (C_{\mathcal{D}} + 1) S^B_{\mathcal{D}}(\overline{u}), \tag{3.8}$$

$$\|\mathcal{H}^B_{\mathcal{D}} u_{\mathcal{D}} - \mathcal{H}^B\overline{u}\| \le W^B_{\mathcal{D}}(\mathcal{H}\overline{u}) + 2 S^B_{\mathcal{D}}(\overline{u}). \tag{3.9}$$

(Note that $\mathcal{H}\overline{u} \in H^B(\Omega)$ because $\mathcal{H}\overline{u} \in L^2(\Omega)^{d\times d}$ and $\mathcal{H} : B^\tau B\mathcal{H}\overline{u} = \mathcal{H} : A\mathcal{H}\overline{u} = f \in L^2(\Omega)$.)

The following convergence result is a trivial consequence of the error estimates above.

Corollary 3.7 (Convergence)**.** *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of B–Hessian discretisations that is coercive, consistent and limit-conforming. Then, as $m \to \infty$, $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} \to \overline{u}$ in $L^2(\Omega)$, $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \to \nabla\overline{u}$ in $L^2(\Omega)^d$ and $\mathcal{H}^B_{\mathcal{D}_m} u_{\mathcal{D}_m} \to \mathcal{H}^B\overline{u}$ in $L^2(\Omega)^{d\times d}$.*

Let us now prove Theorem 3.6.

Proof of Theorem 3.6. For all $v_{\mathcal{D}} \in X_{\mathcal{D},0}$, the equation (2.1a) taken in the sense of distributions shows that $f = \mathcal{H} : A\mathcal{H}\overline{u}$, and thus, by the Hessian scheme (3.1),

$$\int_\Omega \mathcal{H}^B_{\mathcal{D}} u_{\mathcal{D}} : \mathcal{H}^B_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} = \int_\Omega f \Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} = \int_\Omega (\mathcal{H} : B^\tau B\mathcal{H}\overline{u}) \Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x}.$$

Using the definition of $W^B_{\mathcal{D}}$, we infer

$$\int_\Omega \left(\mathcal{H}^B\overline{u} - \mathcal{H}^B_{\mathcal{D}} u_{\mathcal{D}}\right) : \mathcal{H}^B_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} \le W^B_{\mathcal{D}}(\mathcal{H}\overline{u}) \|\mathcal{H}^B_{\mathcal{D}} v_{\mathcal{D}}\|. \tag{3.10}$$

Define the interpolant $P_{\mathcal{D}} : H_0^2(\Omega) \to X_{\mathcal{D},0}$ by

$$P_{\mathcal{D}}\overline{u} = \operatorname*{argmin}_{w\in X_{\mathcal{D},0}} \Big(\|\Pi_{\mathcal{D}} w - \overline{u}\| + \|\nabla_{\mathcal{D}} w - \nabla \overline{u}\| + \|\mathcal{H}_{\mathcal{D}}^B w - \mathcal{H}^B \overline{u}\| \Big)$$

and notice that

$$\|\Pi_{\mathcal{D}} P_{\mathcal{D}}\overline{u} - \overline{u}\| + \|\nabla_{\mathcal{D}} P_{\mathcal{D}}\overline{u} - \nabla \overline{u}\| + \|\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}^B \overline{u}\| \le S_{\mathcal{D}}^B(\overline{u}). \tag{3.11}$$

Introducing the term $\mathcal{H}^B\overline{u}$ and using (3.10), we obtain

$$\begin{aligned}
&\int_\Omega \Big(\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}}\Big) : \mathcal{H}_{\mathcal{D}}^B v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} \\
&\quad = \int_\Omega \Big(\mathcal{H}^B\overline{u} - \mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}}\Big) : \mathcal{H}_{\mathcal{D}}^B v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} + \int_\Omega \Big(\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}^B\overline{u}\Big) : \mathcal{H}_{\mathcal{D}}^B v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} \\
&\quad \le W_{\mathcal{D}}^B(\mathcal{H}\overline{u})\|\mathcal{H}_{\mathcal{D}}^B v_{\mathcal{D}}\| + \|\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}^B\overline{u}\|\,\|\mathcal{H}_{\mathcal{D}}^B v_{\mathcal{D}}\|.
\end{aligned}$$

Choosing $v_{\mathcal{D}} = P_{\mathcal{D}}\overline{u} - u_{\mathcal{D}}$, we get

$$\begin{aligned}
\|\mathcal{H}_{\mathcal{D}}^B(P_{\mathcal{D}}\overline{u} - u_{\mathcal{D}})\|^2 \le{}& W_{\mathcal{D}}^B(\mathcal{H}\overline{u})\|\mathcal{H}_{\mathcal{D}}^B(P_{\mathcal{D}}\overline{u} - u_{\mathcal{D}})\| \\
&+ \|\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}^B\overline{u}\|\,\|\mathcal{H}_{\mathcal{D}}^B(P_{\mathcal{D}}\overline{u} - u_{\mathcal{D}})\|.
\end{aligned}$$

Thus, by (3.11),

$$\|\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}}\| \le W_{\mathcal{D}}^B(\mathcal{H}\overline{u}) + S_{\mathcal{D}}^B(\overline{u}). \tag{3.12}$$

A use of triangle inequality, (3.11) and (3.12) yields

$$\begin{aligned}
\|\mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}} - \mathcal{H}^B\overline{u}\| &\le \|\mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}} - \mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u}\| + \|\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}^B\overline{u}\| \\
&\le W_{\mathcal{D}}^B(\mathcal{H}\overline{u}) + 2S_{\mathcal{D}}^B(\overline{u}),
\end{aligned}$$

which is (3.9). Using the definition of $C_{\mathcal{D}}$, and (3.11) and (3.12), we obtain

$$\begin{aligned}
\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\| &\le \|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \Pi_{\mathcal{D}} P_{\mathcal{D}}\overline{u}\| + \|\Pi_{\mathcal{D}} P_{\mathcal{D}}\overline{u} - \overline{u}\| \\
&\le C_{\mathcal{D}}\|\mathcal{H}_{\mathcal{D}}^B P_{\mathcal{D}}\overline{u} - \mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}}\| + S_{\mathcal{D}}^B(\overline{u}) \\
&\le C_{\mathcal{D}} W_{\mathcal{D}}^B(\mathcal{H}\overline{u}) + (C_{\mathcal{D}} + 1)S_{\mathcal{D}}^B(\overline{u}).
\end{aligned}$$

Hence, (3.7) is established, and (3.8) follows in a similar way. □

We now aim to present particular HDMs. The first (in Section 4) is a novel scheme based on gradient recovery operators, and a particular cheap construction of these operators using biorthogonal basis. Then, we show that a finite volume method (in Section 5) and known finite element methods (in Section 7) fit into the HDM. Let us first set some notations related to meshes.

Definition 3.8 (Polytopal mesh [10, Definition 7.2])**.** *Let Ω be a bounded polytopal open subset of $\mathbb{R}^d$ ($d \ge 1$). A polytopal mesh of Ω is $\mathcal{T} = (\mathcal{M}, \mathcal{F}, \mathcal{P})$, where:*

(1) $\mathcal{M}$ is a finite family of non empty connected polytopal open disjoint subsets of Ω (the cells) such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any $K \in \mathcal{M}$, $|K| > 0$ is the measure of K, h_K denotes the diameter of K, $\overline{\boldsymbol{x}}_K$ is the center of mass of K, and n_K is the outer unit normal to K.

(2) $\mathcal{F}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the edges of the mesh in 2D, the faces in 3D), such that any $\sigma \in \mathcal{F}$ is a non empty open subset of a hyperplane of $\mathbb{R}^d$ and $\sigma \subset \overline{\Omega}$. Assume that for all $K \in \mathcal{M}$ there exists a subset $\mathcal{F}_K$ of $\mathcal{F}$ such that the boundary of K is $\bigcup_{\sigma\in\mathcal{F}_K}\overline{\sigma}$. We then set $\mathcal{M}_\sigma = \{K \in \mathcal{M}\,;\, \sigma \in \mathcal{F}_K\}$ and assume that, for all $\sigma \in \mathcal{F}$, $\mathcal{M}_\sigma$ has exactly

one element and $\sigma \subset \partial\Omega$, or $\mathcal{M}_\sigma$ has two elements and $\sigma \subset \Omega$. Let $\mathcal{F}_{\rm int}$ be the set of all interior faces, i.e. $\sigma \in \mathcal{F}$ such that $\sigma \subset \Omega$, and $\mathcal{F}_{\rm ext}$ the set of boundary faces, i.e. $\sigma \in \mathcal{F}$ such that $\sigma \subset \partial\Omega$. The $(d-1)$-dimensional measure of $\sigma \in \mathcal{F}$ is $|\sigma|$, and its centre of mass is $\overline{\boldsymbol{x}}_\sigma$.

(3) *$\mathcal{P} = (\boldsymbol{x}_K)_{K\in\mathcal{M}}$ is a family of points of Ω indexed by $\mathcal{M}$ and such that, for all $K \in \mathcal{M}$, $\boldsymbol{x}_K \in K$. Assume that any cell $K \in \mathcal{M}$ is strictly $\boldsymbol{x}_K$-star-shaped, meaning that if $\boldsymbol{x} \in \overline{K}$ then the line segment $[\boldsymbol{x}_K, \boldsymbol{x})$ is included in K.*

The diameter of such a polytopal mesh is $h = \max_{K\in\mathcal{M}} h_K$.

4. Method based on Gradient Recovery Operators

4.1. **General setting.** Let V_h be an H^1_0-conforming finite element space with underlying mesh $\mathcal{M} = \mathcal{M}_h$. We assume that V_h contains the piecewise linear functions, and that $\mathcal{M}_h$ satisfies usual regularity assumptions, namely, denoting by $\rho_K = \max\{r > 0\,;\, B(\overline{\boldsymbol{x}}_K, r) \subset K\}$ the maximal radius of balls centred at $\overline{\boldsymbol{x}}_K$ and included in K, we assume that there exists $\eta > 0$ (independent of h) such that

$$\forall K \in \mathcal{M}\,,\ \eta \ge \frac{h_K}{\rho_K}. \tag{4.1}$$

The gradient ∇u of $u \in V_h$ is well defined, but its second derivative $\nabla\nabla u$ is not. In order to compute some sort of second derivatives, consider a projector $Q_h : L^2(\Omega) \to V_h$, which is extended to $L^2(\Omega)^d$ component-wise. Then ∇u can be projected onto V_h^d, and the resulting function $Q_h\nabla u \in V_h^d$ is differentiable. We can then consider $\nabla(Q_h\nabla u)$ as a sort of Hessian of u. However, it not necessarily clear, for some interesting choices of practically computable Q_h (see Section 4.2), that this reconstructed Hessian has proper coercivity properties. We therefore also consider a function $\mathfrak{S}_h$ whose role is to stabilise this reconstructed Hessian.
Let $(V_h, Q_h, I_h, \mathfrak{S}_h)$ be a quadruplet of a finite element space $V_h \subset H^1_0(\Omega)$, a reconstruction operator $Q_h : L^2(\Omega) \to V_h$ that is a projector onto V_h (that is, $Q_h = \mathrm{Id}$ on V_h), an interpolant $I_h : H^2_0(\Omega) \to V_h$ and a stabilisation function $\mathfrak{S}_h \in L^\infty(\Omega)^d$ such that, with constants C not depending on h,

- **(P0)** [*Strucure of V_h and I_h*] The inverse estimate $\|\nabla z\| \le Ch^{-1}\|z\|$ holds for all $z \in V_h$ and, for $\varphi \in H^2_0(\Omega)$, we have $\|\nabla I_h\varphi - \nabla\varphi\| \le Ch\|\varphi\|_{H^2(\Omega)}$.
- **(P1)** [*Stability of Q_h*] For $\phi \in L^2(\Omega)$, we have $\|Q_h\phi\| \le C\|\phi\|$.
- **(P2)** [*$Q_h\nabla I_h$ approximates ∇*] For some space W densely embedded in $H^3(\Omega) \cap H^2_0(\Omega)$ and for all $\psi \in W$, we have $\|Q_h\nabla I_h\psi - \nabla\psi\| \le Ch^2\|\psi\|_W$.
- **(P3)** [*H^1 approximation property of Q_h*] For $w \in H^2(\Omega) \cap H^1_0(\Omega)$, we have $\|\nabla Q_h w - \nabla w\| \le Ch\|w\|_{H^2(\Omega)}$.
- **(P4)** [*Asymptotic density of $[(Q_h\nabla - \nabla)(V_h)]^\perp$*] Setting $N_h = [(Q_h\nabla - \nabla)(V_h)]^\perp$, where the orthogonality is considered for the $L^2(\Omega)^d$-inner product, the following approximation property holds:
$$\inf_{\mu_h\in N_h} \|\mu_h - \varphi\| \le Ch\|\varphi\|_{H^1(\Omega)^d},\ \forall\varphi \in H^1(\Omega)^d,$$
- **(P5)** [*Stabilisation function*] $1 \le |\mathfrak{S}_h| \le C$ and, for all $K \in \mathcal{M}$, denoting by $V_h(K) = \{v_{|K}\,;\, v \in V_h\,,\ K \in \mathcal{M}\}$ the local FE space,
$$\big[\mathfrak{S}_{h|K} \otimes (Q_h\nabla - \nabla)(V_h(K))\big] \perp \nabla V_h(K)^d,$$

where the orthogonality is understood in $L^2(K)^{d\times d}$ with the inner product induced by ":".

Remark 4.1. *A classical operator Q_h that satisfies these assumptions, for standard FE spaces V_h, is the L^2-orthogonal projector on V_h. This operator is however non-local and complicated to compute. We present in Section 4.2 a much more efficient construction of Q_h, local and based on biorthogonal bases.*

To construct an HD based on such a quadruplet, we assume the following stronger form of (2.3):

$$\exists C_B > 0 \,:\, |B\boldsymbol{\xi}| \geq C_B|\boldsymbol{\xi}|\,, \quad \forall \boldsymbol{\xi} \in \mathcal{S}_d(\mathbb{R}). \tag{4.2}$$

Definition 4.2 (B–Hessian discretisation using gradient recovery)**.** *Under Assumption (4.2), the B-Hessian discretisation based on a quadruplet $(V_h, Q_h, I_h, \mathfrak{S}_h)$ satisfying* **(P0)**–**(P5)** *is defined by: $X_{\mathcal{D},0} = V_h$ and, for $u \in X_{\mathcal{D},0}$,*

$$\Pi_{\mathcal{D}} u = u\,,\ \nabla_{\mathcal{D}} u = Q_h\nabla u \text{ and } \mathcal{H}_{\mathcal{D}}^B u = B\left[\nabla(Q_h\nabla u) + \mathfrak{S}_h \otimes (Q_h\nabla u - \nabla u)\right].$$

The next theorem gives an estimate on the accuracy measures $C_{\mathcal{D}}^B$, $S_{\mathcal{D}}^B$ and $W_{\mathcal{D}}^B$ associated with an HD $\mathcal{D}$ using gradient recovery. Incidentally, the estimate on $C_{\mathcal{D}}^B$ also establishes that $\|\mathcal{H}_{\mathcal{D}}^B \cdot\|$ is a norm on $X_{\mathcal{D},0}$.

Theorem 4.3 (Estimates for Hessian discretisations based on gradient recovery)**.** *Let $\mathcal{D}$ be a B–Hessian discretisation in the sense of Definition 4.2, with B satisfying Estimate (4.2) and $(V_h, I_h, Q_h, \mathfrak{S}_h)$ satisfying* **(P0)**–**(P5)**. *Then, there exists a constant C, not depending on h, such that*
- *$C_{\mathcal{D}}^B \leq C$,*
- *$\forall\, \varphi \in W$, $S_{\mathcal{D}}^B(\varphi) \leq Ch\|\varphi\|_W$,*
- *$\forall\, \xi \in H^2(\Omega)^{d\times d}$, $W_{\mathcal{D}}^B(\xi) \leq Ch\|\xi\|_{H^2(\Omega)^{d\times d}}$.*

Before proving this theorem, let us note the following straightforward consequence of Remark 3.4.

Corollary 4.4 (Properties of Hessian discretisation based on gradient recovery)**.** *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of B–Hessian discretisations, with B satisfying Estimate (4.2) and each $\mathcal{D}_m$ associated with $(V_{h_m}, Q_{h_m}, I_{h_m}, \mathfrak{S}_{h_m})$ satisfying* **(P0)**–**(P5)** *uniformly with respect to m. Assume that $h_m \to 0$ as $m \to \infty$. Then the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is coercive, consistent and limit-conforming.*

Proof of Theorem 4.3.
• Coercivity: Let $v \in X_{\mathcal{D},0}$. Noticing that $|a \otimes b| = |a||b|$ for any two vectors a and b, the definition of $\mathcal{H}_{\mathcal{D}}^B$, Property (4.2) of B and $|\mathfrak{S}| \geq 1$ yield

$$\begin{aligned}
\|\mathcal{H}_{\mathcal{D}}^B v\|^2 &\geq C_B^2 \int_\Omega |\nabla(Q_h\nabla v) + \mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)|^2 \,\mathrm{d}\boldsymbol{x}\\
&= C_B^2 \int_\Omega |\nabla(Q_h\nabla v)|^2 \,\mathrm{d}\boldsymbol{x} + C_B^2 \int_\Omega |\mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)|^2 \,\mathrm{d}\boldsymbol{x}\\
&\quad + 2C_B^2 \int_\Omega \nabla(Q_h\nabla v) : \mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)\,\mathrm{d}\boldsymbol{x}\\
&\geq C_B^2 \left(\|\nabla(Q_h\nabla v)\|^2 + \|Q_h\nabla v - \nabla v\|^2\right)\\
&\quad + 2C_B^2 \sum_{K\in\mathcal{M}} \int_K \nabla(Q_h\nabla v) : \mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

Since $\nabla(Q_h\nabla v)_{|K} \in \nabla V_h(K)^d$, a use of property **(P5)** shows that the last term vanishes, and we have thus

$$\|\mathcal{H}_{\mathcal{D}}^B v\|^2 \ge C_B^2 \left(\|\nabla(Q_h\nabla v)\|^2 + \|Q_h\nabla v - \nabla v\|^2\right), \tag{4.3}$$

which implies

$$C_B^{-1}\sqrt{2}\|\mathcal{H}_{\mathcal{D}}^B v\| \ge \|\nabla(Q_h\nabla v)\| + \|Q_h\nabla v - \nabla v\|. \tag{4.4}$$

Apply now the Poincaré inequality twice, the triangle inequality and (4.4) to obtain

$$\begin{aligned}\|\Pi_{\mathcal{D}} v\| = \|v\| &\le \operatorname{diam}(\Omega)\|\nabla v\| \\ &\le \operatorname{diam}(\Omega)\|\nabla v - Q_h\nabla v\| + \operatorname{diam}(\Omega)\|Q_h\nabla v\| \\ &\le \operatorname{diam}(\Omega)\|\nabla v - Q_h\nabla v\| + \operatorname{diam}(\Omega)^2\|\nabla(Q_h\nabla v)\| \\ &\le C_B^{-1}\sqrt{2}\max(\operatorname{diam}(\Omega), \operatorname{diam}(\Omega)^2)\|\mathcal{H}_{\mathcal{D}}^B v\|. \end{aligned}\tag{4.5}$$

From (4.3) and the Poincaré inequality, we also have

$$\|\nabla_{\mathcal{D}} v\| = \|Q_h\nabla v\| \le \operatorname{diam}(\Omega)\|\nabla(Q_h\nabla v)\| \le \operatorname{diam}(\Omega)C_B^{-1}\|\mathcal{H}_{\mathcal{D}}^B v\|. \tag{4.6}$$

Estimates (4.5) and (4.6) show that $C_{\mathcal{D}}^B \le C_B^{-1}\sqrt{2}\max(\operatorname{diam}(\Omega), \operatorname{diam}(\Omega)^2)$.

• Consistency: let $\varphi \in W \subset H^3(\Omega) \cap H_0^2(\Omega)$ and choose $v = I_h\varphi \in X_{\mathcal{D},0}$. Using the properties **(P0)** (which implies $\|I_h\varphi - \varphi\| \le Ch\|\varphi\|_{H^2(\Omega)}$ by the Poincaré inequality) and **(P2)**, we obtain

$$\|\Pi_{\mathcal{D}} v - \varphi\| = \|I_h\varphi - \varphi\| \le Ch\|\varphi\|_{H^2(\Omega)} \tag{4.7}$$

and

$$\|\nabla_{\mathcal{D}} v - \nabla\varphi\| = \|Q_h\nabla I_h\varphi - \nabla\varphi\| \le Ch^2\|\varphi\|_W. \tag{4.8}$$

Let us now turn to $\|\mathcal{H}_{\mathcal{D}}^B v - \mathcal{H}^B\varphi\|$. Observe that $\nabla\nabla$ is another notation for $\mathcal{H}$. Using a triangle inequality, the boundedness of B and $\mathfrak{S}_h$ implies

$$\begin{aligned}\|\mathcal{H}_{\mathcal{D}}^B v - \mathcal{H}^B\varphi\| &= \|B\left[\nabla(Q_h\nabla v) + \mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)\right] - B\mathcal{H}\varphi\| \\ &\le \|B\left[\nabla(Q_h\nabla v) - \nabla\nabla\varphi\right]\| + \|B\mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)\| \\ &\le C\underbrace{\|\nabla(Q_h\nabla v) - \nabla\nabla\varphi\|}_{A_1} + C\underbrace{\|Q_h\nabla v - \nabla v\|}_{A_2}. \end{aligned}\tag{4.9}$$

Introducing the term $\nabla(Q_h\nabla\varphi)$, using in sequence the triangle inequality, the inverse inequality in **(P0)**, **(P3)**, the projection property of Q_h, **(P1)** and **(P2)**, we get

$$\begin{aligned} A_1 &\le \|\nabla[Q_h\nabla v - Q_h\nabla\varphi]\| + \|\nabla(Q_h\nabla\varphi) - \nabla\nabla\varphi\| \\ &\le Ch^{-1}\|Q_h\nabla v - Q_h\nabla\varphi\| + Ch\|\nabla\varphi\|_{H^2(\Omega)} \\ &\le Ch^{-1}\|Q_h\left(Q_h\nabla v - \nabla\varphi\right)\| + Ch\|\nabla\varphi\|_{H^2(\Omega)} \\ &\le Ch^{-1}\|Q_h\nabla I_h\varphi - \nabla\varphi\| + Ch\|\nabla\varphi\|_{H^2(\Omega)} \le Ch\|\varphi\|_W. \end{aligned}\tag{4.10}$$

To estimate A_2, we use the properties **(P2)** and **(P0)**:

$$A_2 \le \|Q_h\nabla v - \nabla\varphi\| + \|\nabla\varphi - \nabla v\| \le Ch^2\|\varphi\|_W + Ch\|\varphi\|_{H^2(\Omega)}. \tag{4.11}$$

The estimate on $S_{\mathcal{D}}^B(\varphi)$ follows from (4.7)–(4.11).

• LIMIT-CONFORMITY: for $\xi \in H^2(\Omega)^{d\times d}$ and $v \in X_{\mathcal{D},0}$,

$$\int_\Omega \Big((\mathcal{H} : B^\tau B\xi)\Pi_{\mathcal{D}} v - B\xi : \mathcal{H}_{\mathcal{D}}^B v\Big)\,\mathrm{d}\boldsymbol{x}$$

$$= \underbrace{\int_\Omega \Big((\mathcal{H} : B^\tau B\xi)\Pi_{\mathcal{D}} v - B\xi : B\nabla(Q_h\nabla v)\Big)\,\mathrm{d}\boldsymbol{x}}_{B_1}$$

$$- \underbrace{\int_\Omega B\xi : B\mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)\,\mathrm{d}\boldsymbol{x}}_{B_2}. \tag{4.12}$$

Recall that $v = \Pi_{\mathcal{D}} v$ and $A = B^\tau B$. Since $Q_h\nabla v \in H_0^1(\Omega)$, Lemma A.2 applied to $(\mathcal{H} : A\xi)v$ and an integration-by-parts on $B\xi : B\nabla(Q_h\nabla v) = A\xi : \nabla(Q_h\nabla v)$ show that, for any $\mu_h \in N_h = [(Q_h\nabla - \nabla)(V_h)]^\perp$,

$$\begin{aligned}
|B_1| &= \Big| \int_\Omega (\mathcal{H} : A\xi) v\,\mathrm{d}\boldsymbol{x} + \int_\Omega Q_h\nabla v \cdot \mathrm{div}(A\xi)\,\mathrm{d}\boldsymbol{x}\Big| \\
&= \Big| \int_\Omega (Q_h\nabla v - \nabla v)\cdot \mathrm{div}(A\xi)\,\mathrm{d}\boldsymbol{x}\Big| \\
&= \Big| \int_\Omega (Q_h\nabla v - \nabla v)\cdot (\mathrm{div}(A\xi) - \mu_h)\,\mathrm{d}\boldsymbol{x}\Big| \\
&\le \|Q_h\nabla v - \nabla v\|\,\|\mathrm{div}(A\xi) - \mu_h\|.
\end{aligned} \tag{4.13}$$

Take the infimum over all $\mu_h \in N_h$. Estimate (4.4) and Property **(P4)** yield

$$|B_1| \le Ch\|\mathcal{H}_{\mathcal{D}}^B v\|\,\|\mathrm{div}(A\xi)\|_{H^1(\Omega)^d}. \tag{4.14}$$

Let ξ_K denote the average of ξ over $K \in \mathcal{M}$. By the mesh regularity assumption, $\|\xi - \xi_K\|_{L^2(K)^{d\times d}} \le Ch\|\xi\|_{H^1(K)^{d\times d}}$ (see, e.g., [10, Lemma B.6]). Moreover, since V_h contains the piecewise constant functions, $\nabla V_h(K)$ contains the constant vector-valued functions on K and thus, by the orthogonality condition in **(P5)**, the Cauchy-Schwarz inequality, the boundedness of B and $\mathfrak{S}_h$, and (4.4),

$$\begin{aligned}
|B_2| &= \Big| \sum_{K\in\mathcal{M}} \int_K B^\tau B\xi : \mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)\,\mathrm{d}\boldsymbol{x}\Big| \\
&= \Big| \sum_{K\in\mathcal{M}} \int_K (B^\tau B\xi - B^\tau B\xi_K) : \mathfrak{S}_h \otimes (Q_h\nabla v - \nabla v)\,\mathrm{d}\boldsymbol{x}\Big| \\
&\le C \sum_{K\in\mathcal{M}} \|\xi - \xi_K\|_{L^2(K)} \|Q_h\nabla v - \nabla v\|_{L^2(K)} \\
&\le Ch\|\xi\|_{H^1(\Omega)^{d\times d}}\|\mathcal{H}_{\mathcal{D}}^B v\|.
\end{aligned} \tag{4.15}$$

Plugging (4.14) and (4.15) into (4.12) yields

$$\Big| \int_\Omega \Big((\mathcal{H} : B^\tau B\xi)\Pi_{\mathcal{D}} v - B\xi : \mathcal{H}_{\mathcal{D}}^B v\Big)\,\mathrm{d}\boldsymbol{x}\Big|$$

$$\le Ch\Big(\|\mathrm{div}(A\xi)\|_{H^1(\Omega)^d} + \|\xi\|_{H^1(\Omega)^{d\times d}}\Big)\|\mathcal{H}_{\mathcal{D}}^B v\|.$$

By the definition (3.4) of $W_{\mathcal{D}}^B(\xi)$, this concludes the proof of the estimate on this quantity. □

4.2. **A gradient recovery operator based on biorthogonal systems.** We present here a particular case of a method based on a gradient recovery operator, using biorthogonal systems as in [21]. V_h is the conforming $\mathbb{P}_1$ FE space on a mesh of simplices, and I_h is the Lagrange interpolation with respect to vertices of $\mathcal{M}$. We will build a locally computable projector Q_h, that is, such that determining $Q_h f$ on a cell K only requires the knowledge of f on K and its neighbouring cells. Let $\mathcal{B}_1 := \{\phi_1, \cdots, \phi_n\}$ be the set of basis functions of V_h associated with the inner vertices in $\mathcal{M}$. Let the set $\mathcal{B}_2 := \{\psi_1, \cdots, \psi_n\}$ be the set of discontinuous piecewise linear functions biorthogonal to $\mathcal{B}_1$ also associated with the inner vertices of $\mathcal{M}$, so that elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy the biorthogonality relation

$$\int_\Omega \psi_i \phi_j \,\mathrm{d}\boldsymbol{x} = c_j \delta_{ij},\ c_j \neq 0,\ 1 \leq i, j \leq n, \tag{4.16}$$

where δ_{ij} is the Kronecker symbol and $c_j = \int_\Omega \psi_j \phi_j \, d\boldsymbol{x}$. Let $M_h := \mathrm{span}\{\mathcal{B}_2\}$. Such biorthogonal systems have been constructed in the context of mortar finite elements, and later extended to gradient recovery operators [16, 18, 21]. The basis functions of M_h can be defined on a reference element. For example, for the reference triangle, we have

$$\widehat{\psi}_1(\boldsymbol{x}) := 3 - 4x_1 - 4x_2, \quad \widehat{\psi}_2(\boldsymbol{x}) := 4x_1 - 1, \quad \text{and} \quad \widehat{\psi}_3(\boldsymbol{x}) := 4x_2 - 1,$$

associated with its three vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, respectively. For the reference tetrahedron, we have

$$\begin{aligned} &\widehat{\psi}_1(\boldsymbol{x}) := 4 - 5x_1 - 5x_2 - 5x_3, \quad \widehat{\psi}_2(\boldsymbol{x}) := 5x_1 - 1, \\ &\widehat{\psi}_3(\boldsymbol{x}) := 5x_2 - 1, \quad \text{and} \quad \widehat{\psi}_4(\boldsymbol{x}) := 5x_3 - 1, \end{aligned}$$

associated with its four vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. These basis functions satisfy

$$\sum_{i=1}^{d+1} \widehat{\psi}_i = 1. \tag{4.17}$$

The projection operator $Q_h : L^2(\Omega) \to V_h$ is the oblique projector onto V_h defined as: for $f \in L^2(\Omega)$, $Q_h f \in V_h$ satisfies

$$\int_\Omega (Q_h f)\, \psi_h \,\mathrm{d}\boldsymbol{x} = \int_\Omega f\, \psi_h \,\mathrm{d}\boldsymbol{x}, \quad \forall \psi_h \in M_h. \tag{4.18}$$

Due to the biorthogonality relation (4.16), Q_h is well-defined and has the explicit representation

$$Q_h f = \sum_{i=1}^{n} \frac{\int_\Omega \psi_i\, f \,\mathrm{d}\boldsymbol{x}}{c_i} \phi_i. \tag{4.19}$$

The relation (4.18) shows $M_h \subset [(Q_h - I)(L^2(\Omega))]^\perp$. Hence, if M_h satisfies the approximation property

$$\inf_{\alpha_h \in M_h} \|\alpha_h - \psi\| \leq Ch \|\psi\|_{H^1(\Omega)}, \quad \forall \psi \in H^1(\Omega),$$

we know that **(P4)** holds. In order to get this approximation property it is sufficient that the basis functions of M_h reproduce constant functions. Let $K \in \mathcal{M}$ be an interior element not touching any boundary vertex. Due to the property (4.17)

$$\sum_{i=1}^{d+1} \psi_{\mathsf{v}_i} = 1 \quad \text{on} \quad K,$$

where $\{\psi_{\mathsf{v}_i}\}_{i=1}^{d+1}$ are basis functions of M_h associated with the vertices $(\mathsf{v}_1,\ldots,\mathsf{v}_{d+1})$ of K.

However, this property does not hold on $K\in\mathcal{M}$ if K has one or more vertices on the boundary. We need to modify the piecewise linear basis functions of M_h to guarantee the approximation property [19, 17]. Let $W_h\subset H^1(\Omega)$ be the lowest order FE space including the basis functions on the boundary vertices of $\mathcal{M}$, and let $\widetilde{M}_h$ the space spanned by the discontinuous basis functions biorthogonal to the basis functions of W_h. M_h is then obtained as a modification of $\widetilde{M}_h$, by moving all vertex basis functions of this latter space to nearby internal vertices using the following three steps.

(1) For a basis function $\widetilde{\psi}_k$ of $\widetilde{M}_h$ associated with a vertex v_k on the boundary we find a closest *internal* triangle or tetrahedron $K\in\mathcal{M}$ (that is, K does not have a boundary vertex).
(2) Compute the barycentric coordinates $\{\alpha_{K,i}\}_{i=1}^{d+1}$ of v_k with respect to the vertices of K, and modify all the basis functions $\{\widetilde{\psi}_{K,i}\}_{i=1}^{d+1}$ of $\widetilde{M}_h$ associated with K into $\psi_{K,i}=\widetilde{\psi}_{K,i}+\alpha_{K,i}\widetilde{\psi}_k$ for $i=1,\cdots,d+1$.
(3) Remove $\widetilde{\psi}_k$ from the basis of $\widetilde{M}_h$.

An alternative way is to modify the basis functions of all triangles or tetrahedra having one or more boundary vertices as proposed in [16].

(1) If all vertices $\{\mathsf{v}_i\}_{i=1}^{d+1}$ of an element $K\in\mathcal{M}$ are inner vertices, then the linear basis functions $\{\psi_{\mathsf{v}_i}\}_{i=1}^{d+1}$ of M_h on K are defined using the biorthogonal relationship (4.16) with the basis functions $\{\phi_{\mathsf{v}_i}\}_{i=1}^{d+1}$ of V_h.
(2) If an element $K\in\mathcal{M}$ has all boundary vertices, then we find a neighbouring element $\widetilde{K}$, which has at least one inner vertex v, and we extend the support of the basis function $\psi_{\mathsf{v}}\in M_h$ associated with v to the element K by defining $\psi_{\mathsf{v}}=1$ on K.
(3) If an element $K\in\mathcal{M}$ has only one inner vertex v and other boundary vertices, then the basis function $\psi_{\mathsf{v}}\in M_h$ associated with the inner vertex v is defined as $\psi_{\mathsf{v}}=1$ on K.
(4) If an element K has two inner vertices v_1 and v_2 and other boundary vertices, then the basis functions ψ_{v_1}, $\psi_{\mathsf{v}_2}\in M_h$ associated with these points are chosen to satisfy the biorthogonal relationship (4.16) with ϕ_{v_1}, $\phi_{\mathsf{v}_2}\in V_h$, as well as the property $\psi_{\mathsf{v}_1}+\psi_{\mathsf{v}_2}=1$ on K.
(5) In the three-dimensional case, we can have an element K with three inner vertices $\{\mathsf{v}_i\}_{i=1}^{3}$ and one boundary vertex. In this case we define three basis functions $\{\psi_{\mathsf{v}_i}\}_{i=1}^{3}$ to satisfy the biorthogonal relationship (4.16) with $\{\phi_{\mathsf{v}_i}\}_{i=1}^{3}$ as well as the condition $\sum_{i=1}^{3}\psi_{\mathsf{v}_i}=1$ on K.

The projection Q_h is stable in L^2 and H^1-norms [18], and hence assumption **(P1)** follows. To establish **(P2)**, we need the following mesh assumption.

(M) For any vertex v, denoting by $\mathcal{M}_{\mathsf{v}}$ the set of cells having v as a vertex,

$$\sum_{K\in\mathcal{M}_{\mathsf{v}}}\frac{|K|}{|S_{\mathsf{v}}|}(\overline{\boldsymbol{x}}_K-\mathsf{v})=O(h^2),$$

where S_{v} is the support of the basis function ϕ_{v} of V_h associated with v.

This assumption is satisfied if the triangles of the mesh can be paired in sets of two that share a common edge and form an $O(h^2)$-parallelogram, that is, the lengths of any two opposite edges differ only by $O(h^2)$. In three dimensions, **(M)** is satisfied if the lengths of each pair of opposite edges of a given element are allowed to differ only by $O(h^2)$ [5]. The following theorem establishes **(P2)** with $W = W^{3,\infty}(\Omega) \cap H^2_0(\Omega)$ and can be proved as in [30, 18].

Theorem 4.5. *Let $u \in W^{3,\infty}(\Omega) \cap H^2_0(\Omega)$. Assume that the triangulation satisfies the assumption* **(M)**. *Then*

$$\|Q_h \nabla I_h u - \nabla u\| \le Ch^2 \|u\|_{W^{3,\infty}(\Omega)}.$$

Since Q_h is a projection onto V_h, $Q_h I_h = I_h$. Hence, for $w \in H^2(\Omega) \cap H^1_0(\Omega)$, introducing $Q_h I_h w = I_h w$ and invoking the H^1-stability property of Q_h [17, Lemma 1.8] leads to

$$\|\nabla Q_h w - \nabla w\| \le \|\nabla Q_h(w - I_h w)\| + \|\nabla I_h w - \nabla w\| \le C\|\nabla I_h w - \nabla w\|.$$

The standard approximation properties of V_h then guarantee **(P3)**. The Assumption **(P4)** is satisfied since $M_h \subset N_h$ (M_h is obtained by combining functions in $\widetilde{M}_h$, that satisfies this property) and the basis functions of M_h locally reproduce constant functions. To build $\mathfrak{S}_h$ that satisfies **(P5)**, divide each triangle $K \in \mathcal{M}$ into four equal triangles using the mid-points of each side, and define $\mathfrak{S}_h$ as a piecewise constant function as described in Figure 1. It can be checked that this function satisfies **(P5)**. A similar construction also works on tetrahedra (in which case $\mathfrak{S}_{h|K}$ is equal to 1 on the four sub-tetrahedra constructed around the vertices of K, and -4 in the rest of K).

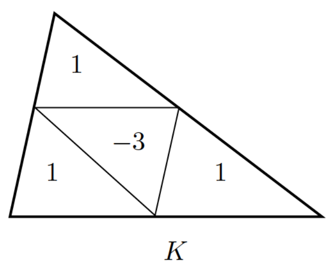

FIGURE 1. Values of the stabilisation function $\mathfrak{S}_h$ inside a cell K.

5. Finite volume method based on Δ-adapted discretizations

We consider here the finite volume (FV) scheme from [12] for the biharmonic problem (2.4) on Δ-adapted meshes, that is, meshes that satisfy an orthogonality property.

Definition 5.1 (Δ-adapted FV mesh)**.** *A general mesh $\mathcal{T}$ is Δ-adapted if*

(1) for all $\sigma \in \mathcal{F}_{\rm int}$, denoting by $K, L \in \mathcal{M}$ the cells such that $\mathcal{M}_\sigma = \{K, L\}$, the straight line $(\boldsymbol{x}_K, \boldsymbol{x}_L)$ intersects and is orthogonal to σ,

(2) for all $\sigma \in \mathcal{F}_{\rm ext}$ with $\mathcal{M}_\sigma = \{K\}$, the line orthogonal to σ going through $\boldsymbol{x}_K$ intersects σ.

For such a mesh, we let $D_{K,\sigma}$ be the cone with vertex $\boldsymbol{x}_K$ and basis σ, and $D_\sigma = \bigcup_{K \in \mathcal{M}_\sigma} D_{K,\sigma}$. For each $\sigma \in \mathcal{F}_{\rm int}$, an orientation is chosen by defining one of the two unit normal vectors n_σ, and we denote by K_σ^- and K_σ^+ the two adjacent control volumes such that n_σ is oriented from K_σ^- to K_σ^+. For all $\sigma \in \mathcal{F}_{\rm ext}$, we denote the control volume $K \in \mathcal{M}$ such that $\sigma \in \mathcal{F}_K$ by K_σ and we define n_σ by $n_{K,\sigma}$. We then set

$$d_\sigma = \begin{cases} \mathrm{dist}(\boldsymbol{x}_{K_\sigma^-},\sigma) + \mathrm{dist}(\boldsymbol{x}_{K_\sigma^+},\sigma) & \forall \sigma \in \mathcal{F}_{\rm int} \\ \mathrm{dist}(\boldsymbol{x}_K,\sigma) & \forall \sigma \in \mathcal{F}_{\rm ext}. \end{cases} \tag{5.1}$$

For all $K \in \mathcal{M}$, set $\mathcal{F}_{K,{\rm int}} = \mathcal{F}_K \cap \mathcal{F}_{\rm int}$ and $\mathcal{F}_{K,{\rm ext}} = \mathcal{F}_K \cap \mathcal{F}_{\rm ext}$. Finally, we define the mesh regularity factor by

$$\theta_{\mathcal{T}} = \max\left\{ \max\left(\frac{\mathrm{diam}(K)}{\mathrm{dist}(\boldsymbol{x}_K,\sigma)}, \frac{d_\sigma}{\mathrm{dist}(\boldsymbol{x}_K,\sigma)} \right) ; \, K \in \mathcal{M}, \sigma \in \mathcal{F}_K \right\}.$$

We now define a notion of B–Hessian discretisation for $B = \frac{\mathrm{tr}(\cdot)}{\sqrt{d}}\mathrm{Id}$, in which case (2.2) corresponds to the biharmonic problem (2.4), for which the coercivity property (2.3) holds (see Section 2.1.1).

Definition 5.2 (B–Hessian discretisation based on Δ-adapted discretisation)**.** *Let $B = \frac{\mathrm{tr}(\cdot)}{\sqrt{d}}\mathrm{Id}$ and $\mathcal{T}$ be a Δ-adapted mesh. A B–Hessian discretisation is given by $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}}, \mathcal{H}_{\mathcal{D}}^B)$ where*

- *$X_{\mathcal{D},0}$ is the space of all real families $u_{\mathcal{D}} = (u_K)_{K\in\mathcal{M}}$, such that $u_K = 0$ for all $K \in \mathcal{M}$ with $\mathcal{F}_{K,{\rm ext}} \neq \emptyset$.*
- *For $u_{\mathcal{D}} \in X_{\mathcal{D},0}$, $\Pi_{\mathcal{D}} u_{\mathcal{D}}$ is the piecewise constant function equal to u_K on the cell K.*
- *The discrete gradient $\nabla_{\mathcal{D}} u_{\mathcal{D}}$ is defined by its constant values on the cells:*
$$\nabla_K u_{\mathcal{D}} = \frac{1}{|K|} \sum_{\sigma\in\mathcal{F}_K} \frac{|\sigma|(\delta_{K,\sigma} u_{\mathcal{D}})(\overline{\boldsymbol{x}}_\sigma - \boldsymbol{x}_K)}{d_\sigma}, \tag{5.2}$$
where
$$\delta_{K,\sigma} u_{\mathcal{D}} = \begin{cases} u_L - u_K & \forall\, \sigma \in \mathcal{F}_{K,{\rm int}}\,,\ \mathcal{M}_\sigma = \{K, L\} \\ 0 & \forall\, \sigma \in \mathcal{F}_{K,{\rm ext}}. \end{cases} \tag{5.3}$$
- *The discrete Laplace operator $\Delta_{\mathcal{D}}$ is defined by its constant values on the cells:*
$$\Delta_K u_{\mathcal{D}} = \frac{1}{|K|} \sum_{\sigma\in\mathcal{F}_K} \frac{|\sigma|\delta_{K,\sigma} u_{\mathcal{D}}}{d_\sigma}. \tag{5.4}$$
We then set $\mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}} = \frac{\Delta_{\mathcal{D}} u_{\mathcal{D}}}{\sqrt{d}}\mathrm{Id}$.

For $u_{\mathcal{D}}, v_{\mathcal{D}} \in X_{\mathcal{D},0}$,

$$[u_{\mathcal{D}}, v_{\mathcal{D}}] = \sum_{\sigma\in\mathcal{F}} \frac{|\sigma|\delta_\sigma u_{\mathcal{D}} \delta_\sigma v_{\mathcal{D}}}{d_\sigma} \tag{5.5}$$

defines an inner product on $X_{\mathcal{D},0}$, whose associated norm is denoted by $\|u_{\mathcal{D}}\|_{\mathcal{D}}$. Here δ_σ is given by

$$\delta_\sigma u_{\mathcal{D}} = \begin{cases} u_{K_\sigma^+} - u_{K_\sigma^-} & \forall\, \sigma \in \mathcal{F}_{\rm int} \\ 0 & \forall\, \sigma \in \mathcal{F}_{\rm ext}. \end{cases} \tag{5.6}$$

It can easily be checked that, with this Hessian discretisation, the Hessian scheme (2.2) is the scheme of [12] for the biharmonic equation. Let us examine the properties of this Hessian discretisation.

Theorem 5.3. *Let $\mathcal{D}$ be a B–Hessian discretisation in the sense of Definition 5.2. Then there exists a constant C, depending only on on $\theta \geq \theta_{\mathcal{T}}$, such that*

- $C^B_{\mathcal{D}} \le C$,
- *If* $\varphi \in C^2_c(\Omega)$, $\Delta\varphi \in H^1(\Omega)$ *and* $a > 0$ *is such that* $\mathrm{supp}(\varphi) \subset \{\boldsymbol{x} \in \Omega\,;\, \mathrm{dist}(\boldsymbol{x}, \partial\Omega) > a\}$*, then*

$$S^B_{\mathcal{D}}(\varphi) \le Ch\|\Delta\varphi\|_{H^1(\Omega)} + Ch\|\varphi\|_{C^2(\overline{\Omega})} \times \begin{cases} |\ln(a)|a^{-3/2} & \text{if } d = 2, \\ a^{-5/3} & \text{if } d = 3. \end{cases} \tag{5.7}$$

- *If* $\varphi \in H^2_0(\Omega) \cap C^2(\overline{\Omega})$ *with* $\Delta\varphi \in H^1(\Omega)$*, then*

$$S^B_{\mathcal{D}}(\varphi) \le Ch\|\Delta\varphi\|_{H^1(\Omega)} + C\|\varphi\|_{C^2(\overline{\Omega})} \times \begin{cases} h^{1/4}|\ln(h)| & \text{if } d = 2, \\ h^{3/13} & \text{if } d = 3. \end{cases} \tag{5.8}$$

- $\forall \xi \in H^2(\Omega)^{d\times d}$, $W^B_{\mathcal{D}}(\xi) \le Ch\|\mathrm{tr}(\xi)\|_{H^2(\Omega)}$.

Remark 5.4. *If the solution* $\overline{u}$ *to* (2.4) *belongs to* $H^4(\Omega)\cap H^2_0(\Omega)$*, then* $\overline{u} \in C^2(\overline{\Omega})$ *and* $\Delta\overline{u} \in H^2(\Omega)$*. In that case, Theorems 3.6 and 5.3 provide an* $O(h^{1/4}|\ln(h)|)$ *(in dimension* $d = 2$*) or* $\mathcal{O}(h^{3/13})$ *(in dimension* $d = 3$*) error estimate for the Hessian scheme based on the HD from Definition 5.2. This slightly improves the result of* [12, Theorem 4.3]*, in which an* $O(h^{1/5})$ *estimate is obtained if* $\overline{u} \in C^4(\overline{\Omega}) \cap H^2_0(\Omega)$.

As for the method based on gradient recovery operators, the properties of the Hessian discretisation follow from the estimates in Theorem 5.3 and from Remark 3.4.

Corollary 5.5. *Let* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *be a sequence of B–Hessian discretisations in the sense of Definition 5.2, associated to meshes such that* $h_m \to 0$ *and* $(\theta_{\mathcal{T}_m})_{m\in\mathbb{N}}$ *is bounded. Then the sequence* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *is coercive, consistent and limit-conforming.*
Proof of Theorem 5.3.

• Coercivity: the discrete Poincaré inequality of [11] states that

$$\|\Pi_{\mathcal{D}} v_{\mathcal{D}}\| \le \mathrm{diam}(\Omega)\|v_{\mathcal{D}}\|_{\mathcal{D}}\,, \quad \forall v_{\mathcal{D}} \in X_{\mathcal{D},0}. \tag{5.9}$$

Let us first prove that

$$-\int_\Omega \Pi_{\mathcal{D}} u_{\mathcal{D}} \Delta_{\mathcal{D}} v_{\mathcal{D}} dx = [u_{\mathcal{D}}, v_{\mathcal{D}}]_{\mathcal{D}}, \quad u_{\mathcal{D}},\, v_{\mathcal{D}} \in X_{\mathcal{D},0}. \tag{5.10}$$

The definitions of $\Pi_{\mathcal{D}}$ and $\Delta_{\mathcal{D}}$ yield

$$-\int_\Omega \Pi_{\mathcal{D}} u_{\mathcal{D}} \Delta_{\mathcal{D}} v_{\mathcal{D}} dx = \sum_{K\in\mathcal{M}} -|K|u_K \Delta_K v_{\mathcal{D}} = -\sum_{K\in\mathcal{M}} u_K \sum_{\sigma\in\mathcal{F}_K} \frac{|\sigma|\delta_{K,\sigma} v_{\mathcal{D}}}{d_\sigma}.$$

For $\sigma \in \mathcal{F}_{\mathrm{ext}}$, $\delta_{K,\sigma} v_{\mathcal{D}} = 0$. Gathering the sums by edges and using (5.3) and (5.6), we obtain

$$-\int_\Omega \Pi_{\mathcal{D}} u_{\mathcal{D}} \Delta_{\mathcal{D}} v_{\mathcal{D}} dx = \sum_{K\in\mathcal{M}} u_K \sum_{\sigma\in\mathcal{F}_{K,\mathrm{int}}} \frac{|\sigma|(v_K - v_L)}{d_\sigma} = \sum_{\sigma\in\mathcal{F}_{\mathrm{int}}} \frac{|\sigma|\delta_\sigma u_{\mathcal{D}} \delta_\sigma v_{\mathcal{D}}}{d_\sigma},$$

which establishes (5.10). Choosing $v_{\mathcal{D}} = u_{\mathcal{D}}$, applying the Cauchy–Schwarz inequality and using (5.9), we get

$$\|u_{\mathcal{D}}\|^2_{\mathcal{D}} \le \|\Pi_{\mathcal{D}} u_{\mathcal{D}}\|\,\|\Delta_{\mathcal{D}} u_{\mathcal{D}}\| \le \mathrm{diam}(\Omega)\|u_{\mathcal{D}}\|_{\mathcal{D}}\|\Delta_{\mathcal{D}} u_{\mathcal{D}}\|.$$

Thus,

$$\|u_{\mathcal{D}}\|_{\mathcal{D}} \le \mathrm{diam}(\Omega)\|\Delta_{\mathcal{D}} u_{\mathcal{D}}\|. \tag{5.11}$$

Combining (5.9) and (5.11), we get

$$\|\Pi_{\mathcal{D}} v_{\mathcal{D}}\| \le \mathrm{diam}(\Omega)^2\|\Delta_{\mathcal{D}} u_{\mathcal{D}}\|. \tag{5.12}$$

The stability of the discrete gradient [12, Lemma 4.1] yields

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}\| \le \theta\sqrt{d}\|u_{\mathcal{D}}\|_{\mathcal{D}} \quad \forall u_{\mathcal{D}} \in X_{\mathcal{D},0}.$$

Estimate (5.11) then shows that $\|\nabla_{\mathcal{D}} u_{\mathcal{D}}\| \le \mathrm{diam}(\Omega)\theta\sqrt{d}\|\Delta_{\mathcal{D}} u_{\mathcal{D}}\|$, which, together with (5.12), concludes the proof of the estimate on $C_{\mathcal{D}}^B$.

• CONSISTENCY – COMPACT SUPPORT: The proof utilises the ideas of [12], with a few improvements of the estimates. For $s > 0$ we let $\Omega_s = \{x \in \Omega\,;\, \mathrm{dist}(\boldsymbol{x}, \partial\Omega) > s\}$. In this proof, $A \lesssim B$ means that $A \le CB$ for some constant C depending only on θ.

We first consider the case where $\varphi \in C_c^2(\Omega)$ and $\Delta\varphi \in H^1(\Omega)$, with support at distance from $\partial\Omega$ equal to or greater than a. As in [12, Proof of Lemma 4.4], let $\psi^a \in C_c^\infty(\Omega)$, equal to 1 on $\Omega_{3a/4}$, that vanishes on $\Omega\backslash\Omega_{a/4}$, and such that, for all $\alpha \in \mathbb{N}^d$, with $|\alpha| = \sum_{i=1}^d \alpha_i$,

$$\|\partial^\alpha \psi^a\|_{L^\infty(\Omega)} \lesssim a^{-|\alpha|}. \tag{5.13}$$

Letting $\psi_{\mathcal{D}}^a = (\psi^a(\boldsymbol{x}_K))_{K\in\mathcal{M}}$, we have $|\Delta_{\mathcal{D}}\psi_{\mathcal{D}}^a| \lesssim a^{-2}$. Hence, for all $r \in [1,\infty]$, since $\Omega\backslash\Omega_{2a}$ has measure $\lesssim a$,

$$\|\Delta_{\mathcal{D}}\psi_{\mathcal{D}}^a\|_{L^r(\Omega)} \lesssim a^{-2+\frac{1}{r}}. \tag{5.14}$$

Letting $\widetilde{v} = (\widetilde{v}_K)_{K\in\mathcal{M}}$ be the solution of the two-point flux approximation finite volume scheme with homogeneous Dirichlet boundary conditions and source term $-\Delta\varphi$, by [11] we have, with $\varphi_{\mathcal{D}} = (\varphi(\boldsymbol{x}_K))_{K\in\mathcal{M}}$,

$$\left(\sum_{\sigma\in\mathcal{F}} \frac{|\sigma|}{d_\sigma}(\delta_\sigma(\widetilde{v} - \varphi_{\mathcal{D}}))^2\right)^{1/2} \lesssim h\|\varphi\|_{C^2(\overline{\Omega})} \tag{5.15}$$

and, for $q \in [1,+\infty)$ if $d = 2$, $q \in [1,6]$ if $d = 3$,

$$\left(\sum_{K\in\mathcal{M}} |K|\,|\widetilde{v}_K - \varphi(\boldsymbol{x}_K)|^q\right)^{1/q} \lesssim qh\|\varphi\|_{C^2(\overline{\Omega})}. \tag{5.16}$$

We then set $w = (\psi^a(\boldsymbol{x}_K)\widetilde{v}_K)_{K\in\mathcal{M}}$, that belongs to $X_{\mathcal{D},0}$ if $h \le a/4$. It is proved in [12, Proof of Lemma 4.4, p. 2032] that, with $[\Delta\varphi]_K = \frac{1}{|K|}\int_K \Delta\varphi\,dx$,

$$\begin{aligned}\Delta_K w - [\Delta\varphi]_K &= (\widetilde{v}_K - \varphi(\boldsymbol{x}_K))\Delta_K\psi_{\mathcal{D}}^a + \frac{1}{|K|}\sum_{\sigma\in\mathcal{F}_K}\frac{|\sigma|}{d_\sigma}(\delta_{K,\sigma}\psi_{\mathcal{D}}^a)\delta_{K,\sigma}(\widetilde{v} - \varphi_{\mathcal{D}}),\\ &= T_{1,K} + T_{2,K}.\end{aligned} \tag{5.17}$$

Using Hölder's inequality with exponents $(q, \frac{2q}{q-2})$, for some $q > 2$ admissible in (5.16), and recalling (5.14), we have

$$\left(\sum_{K\in\mathcal{M}} |K|\,|T_{1,K}|^2\right)^{1/2} \lesssim qha^{-2+\frac{q-2}{2q}}\|\varphi\|_{C^2(\overline{\Omega})}. \tag{5.18}$$

On the other hand, we have $|\delta_{K,\sigma}\psi_{\mathcal{D}}^a| \lesssim d_\sigma a^{-1}$ (see [12, Proof of Lemma 4.4]). Hence, by Cauchy–Schwarz inequality on the sum over the faces, and using the estimate $\sum_{\sigma\in\mathcal{F}_K}|\sigma|d_\sigma \lesssim |K|$,

$$|T_{2,K}|^2 \lesssim \frac{a^{-2}}{|K|^2}\left(\sum_{\sigma\in\mathcal{F}_K}|\sigma|\,|\delta_{K,\sigma}(\widetilde{v} - \varphi_{\mathcal{D}})|\right)^2 \lesssim \frac{a^{-2}}{|K|}\sum_{\sigma\in\mathcal{F}_K}\frac{|\sigma|}{d_\sigma}(\delta_{K,\sigma}(\widetilde{v} - \varphi_{\mathcal{D}}))^2.$$

Estimate (5.15) thus leads to

$$\left(\sum_{K\in\mathcal{M}}|K|\,|T_{2,K}|^2\right)^{1/2}\lesssim a^{-1}h\|\varphi\|_{C^2(\overline{\Omega})}. \tag{5.19}$$

Denote by $[\Delta\varphi]_{\mathcal{D}}$ the piecewise constant function equal to $[\Delta\varphi]_K$ on $K\in\mathcal{M}$. Taking the L^2 norm of (5.17) and using (5.18) and (5.19), we arrive at, since $a^{-1}\lesssim a^{-\frac{3}{2}-\frac{1}{q}}$,

$$\|\Delta_{\mathcal{D}}w-[\Delta\varphi]_{\mathcal{D}}\|_{L^2(\Omega)}\lesssim qha^{-\frac{3}{2}-\frac{1}{q}}\|\varphi\|_{C^2(\overline{\Omega})}.$$

Taking $q=|\ln(a)|$ if $d=2$ or $q=6$ if $d=3$ shows that

$$\|\Delta_{\mathcal{D}}w-[\Delta\varphi]_{\mathcal{D}}\|_{L^2(\Omega)}\lesssim h\|\varphi\|_{C^2(\overline{\Omega})}\times\begin{cases}|\ln(a)|a^{-3/2} & \text{if } d=2,\\ a^{-5/3} & \text{if } d=3.\end{cases} \tag{5.20}$$

A classical estimate [10, Lemma B.6] gives

$$\|[\Delta\varphi]_{\mathcal{D}}-\Delta\varphi\|_{L^2(\Omega)}\lesssim h\|\Delta\varphi\|_{H^1(\Omega)}, \tag{5.21}$$

which shows that $\|\Delta_{\mathcal{D}}w-\Delta\varphi\|_{L^2(\Omega)}$ is bounded above by the right-hand side of (5.7). The estimates on $\nabla_{\mathcal{D}}w-\nabla\varphi$ and on $\Pi_{\mathcal{D}}w-\varphi$ follow as in [12, Lemma 4.4].

• Consistency – general case: Consider now $\varphi\in H^2_0(\Omega)\cap C^2(\overline{\Omega})$, and take ψ^a as above. The boundary conditions on φ show that $|\varphi(\boldsymbol{x})|\lesssim\|\varphi\|_{C^2(\overline{\Omega})}\mathrm{dist}(\boldsymbol{x},\partial\Omega)^2$ and $|\nabla\varphi(\boldsymbol{x})|\lesssim\|\varphi\|_{C^2(\overline{\Omega})}\mathrm{dist}(\boldsymbol{x},\partial\Omega)$. Hence, using (5.13), $|\Omega\backslash\Omega_a|\lesssim a$ and the fact that $1-\psi^a=0$ in Ω_a, we see that, for all $\alpha\in\mathbb{N}^d$ with $|\alpha|\le 2$,

$$\|\partial^\alpha\varphi-\partial^\alpha(\psi^a\varphi)\|_{L^2(\Omega)}\lesssim a^{1/2}\|\varphi\|_{C^2(\overline{\Omega})}. \tag{5.22}$$

Since $\Delta=\sum_{i=1}^2\partial_i^2$, the above estimate applies to Δ instead of ∂^α and, as a consequence,

$$\|[\Delta\varphi]_{\mathcal{D}}-[\Delta(\psi^a\varphi)]_{\mathcal{D}}\|_{L^2(\Omega)}\le\|\Delta\varphi-\Delta(\psi^a\varphi)\|_{L^2(\Omega)}\lesssim a^{1/2}\|\varphi\|_{C^2(\overline{\Omega})}. \tag{5.23}$$

Consider now the interpolant $w\in X_{\mathcal{D},0}$ for $\psi^a\varphi\in C^2_c(\Omega)$ constructed above. Applying (5.20) to $\psi^a\varphi$ instead of φ, noting that $\|\psi^a\varphi\|_{C^2(\overline{\Omega})}\lesssim\|\varphi\|_{C^2(\overline{\Omega})}$ (consequence of (5.22)), and using (5.23), we obtain

$$\|\Delta_{\mathcal{D}}w-[\Delta\varphi]_{\mathcal{D}}\|_{L^2(\Omega)}\lesssim a^{1/2}\|\varphi\|_{C^2(\overline{\Omega})}+h\|\varphi\|_{C^2(\overline{\Omega})}\times\begin{cases}|\ln(a)|a^{-3/2} & \text{if } d=2,\\ a^{-5/3} & \text{if } d=3.\end{cases}$$

Taking $a=h^{1/2}$ if $d=2$ or $a=h^{6/13}$ if $d=3$ leads to

$$\|\Delta_{\mathcal{D}}w-[\Delta\varphi]_{\mathcal{D}}\|_{L^2(\Omega)}\lesssim\|\varphi\|_{C^2(\overline{\Omega})}\times\begin{cases}h^{1/4}|\ln(h)| & \text{if } d=2,\\ h^{3/13} & \text{if } d=3.\end{cases}$$

Combined with (5.21) this shows that $\|\Delta_{\mathcal{D}}w-\Delta\varphi\|_{L^2(\Omega)}$ is bounded above by the right-hand side of (5.8). The estimates on $\Pi_{\mathcal{D}}w-\varphi$ and $\nabla_{\mathcal{D}}w-\nabla\varphi$ follow in a similar way.

• Limit-conformity: For $\xi\in H^B(\Omega)$ and $v_{\mathcal{D}}\in X_{\mathcal{D},0}$, $B=\frac{\mathrm{tr}(\cdot)}{\sqrt{d}}\mathrm{Id}$ implies

$$\int_\Omega(\mathcal{H}:B^\tau B\xi)\Pi_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}=\int_\Omega(B\mathcal{H}:B\xi)\Pi_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}=\int_\Omega\Delta\phi\Pi_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x},$$

where $\phi=\mathrm{tr}(\xi)$. Also, by definition of $\mathcal{H}^B_{\mathcal{D}}$,

$$\int_\Omega B\xi:\mathcal{H}^B_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}=\int_\Omega\phi\Delta_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}.$$

Thus, (3.4) can be rewritten as

$$W_{\mathcal{D}}^{B}(\xi) = \max_{v_{\mathcal{D}} \in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|\mathcal{H}_{\mathcal{D}}^{B} v_{\mathcal{D}}\|} \left| \int_{\Omega} \Big(\Delta\phi \Pi_{\mathcal{D}} v_{\mathcal{D}} - \phi \Delta_{\mathcal{D}} v_{\mathcal{D}} \Big) \,\mathrm{d}\boldsymbol{x} \right|, \tag{5.24}$$

where $\phi = \mathrm{tr}(\xi)$. Define

$$\widehat{\delta}_{\sigma}\phi = \begin{cases} \phi(\boldsymbol{x}_{K_{\sigma}^{+}}) - \phi(\boldsymbol{x}_{K_{\sigma}^{-}}) & \forall \sigma \in \mathcal{F}_{\mathrm{int}} \\ \phi(\boldsymbol{z}_{\sigma}) - \phi(\boldsymbol{x}_{K_{\sigma}}) & \forall \sigma \in \mathcal{F}_{\mathrm{ext}}, \end{cases} \tag{5.25}$$

where $\boldsymbol{z}_{\sigma}$ is the orthogonal projection of $\boldsymbol{x}_K$ on the hyperplane which contains σ. For $\xi \in H^2(\Omega)^{d\times d}$, using the divergence theorem,

$$\int_{\Omega} \Delta\phi \Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} \int_{K} \Delta\phi \Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{F}_K} v_K \int_{\sigma} \nabla\phi \cdot n_{K,\sigma} \,\mathrm{d}s(\boldsymbol{x}).$$

Gathering over the edges and using the definition of δ_{σ}, this leads to

$$\begin{aligned} \int_{\Omega} \Delta\phi \Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} &= -\sum_{\sigma\in\mathcal{F}} \delta_{\sigma} v_{\mathcal{D}} \int_{\sigma} \nabla\phi \cdot n_{\sigma} \,\mathrm{d}s(\boldsymbol{x}) \\ &= -\sum_{\sigma\in\mathcal{F}} \delta_{\sigma} v_{\mathcal{D}} \int_{\sigma} \Big(\frac{\widehat{\delta}_{\sigma}\phi}{d_{\sigma}} + \nabla\phi \cdot n_{\sigma} - \frac{\widehat{\delta}_{\sigma}\phi}{d_{\sigma}} \Big) \,\mathrm{d}s(\boldsymbol{x}) \\ &= -\sum_{\sigma\in\mathcal{F}} \delta_{\sigma} v_{\mathcal{D}} \frac{\widehat{\delta}_{\sigma}\phi|\sigma|}{d_{\sigma}} + \sum_{\sigma\in\mathcal{F}} \delta_{\sigma} v_{\mathcal{D}} \int_{\sigma} \Big(\frac{\widehat{\delta}_{\sigma}\phi}{d_{\sigma}} - \nabla\phi \cdot n_{\sigma} \Big) \,\mathrm{d}s(\boldsymbol{x}). \end{aligned} \tag{5.26}$$

Since $\delta_{\sigma} v_{\mathcal{D}} = 0$ for any $\sigma \in \mathcal{F}_{\mathrm{ext}}$, (5.25), (5.3) and (5.4) imply

$$\begin{aligned} -\sum_{\sigma\in\mathcal{F}} \delta_{\sigma} v_{\mathcal{D}} \frac{\widehat{\delta}_{\sigma}\phi|\sigma|}{d_{\sigma}} &= -\sum_{\sigma\in\mathcal{F}_{\mathrm{int}}} \frac{|\sigma|}{d_{\sigma}} \delta_{\sigma} v_{\mathcal{D}} \Big(\phi(\boldsymbol{x}_{K_{\sigma}}^{+}) - \phi(\boldsymbol{x}_{K_{\sigma}}^{-}) \Big) \\ &= \sum_{K\in\mathcal{M}} \phi(\boldsymbol{x}_K) \sum_{\sigma\in\mathcal{F}_K} \frac{|\sigma|}{d_{\sigma}} \delta_{K,\sigma} v_{\mathcal{D}} = \sum_{K\in\mathcal{M}} |K| \phi(\boldsymbol{x}_K) \Delta_K v_{\mathcal{D}}. \end{aligned}$$

Substituting this in (5.26), we obtain

$$\begin{aligned} \int_{\Omega} \Delta\phi \Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} = &\sum_{K\in\mathcal{M}} |K| \phi(\boldsymbol{x}_K) \Delta_K v_{\mathcal{D}} \\ &+ \sum_{\sigma\in\mathcal{F}} \delta_{\sigma} v_{\mathcal{D}} \int_{\sigma} \Big(\frac{\widehat{\delta}_{\sigma}\phi}{d_{\sigma}} - \nabla\phi \cdot n_{\sigma} \Big) \,\mathrm{d}s(\boldsymbol{x}). \end{aligned} \tag{5.27}$$

To deal with the first term, we first combine the two estimates in [10, Lemma 7.61] to see that

$$|\phi(\boldsymbol{x}_K) - \phi(\boldsymbol{y})| \le Ch|K|^{-1/2} \|\phi\|_{H^2(K)}, \qquad \forall \boldsymbol{y} \in K.$$

Hence, using the Cauchy–Schwarz inequality,

$$\begin{aligned}\Bigg|\sum_{K\in\mathcal{M}}|K|\phi(\boldsymbol{x}_K)\Delta_K v_{\mathcal{D}} - \int_\Omega \phi\Delta_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}\Bigg| \\ = \Bigg|\sum_{K\in\mathcal{M}}|K|\left(\phi(\boldsymbol{x}_K)-\frac{1}{|K|}\int_K\phi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\right)\Delta_K v_{\mathcal{D}}\Bigg| \\ \le Ch\|\phi\|_{H^2(\Omega)}\left(\sum_{K\in\mathcal{M}}|K||\Delta_K v_{\mathcal{D}}|^2\right)^{1/2} = Ch\|\phi\|_{H^2(\Omega)}\|\Delta_{\mathcal{D}}v_{\mathcal{D}}\|. \end{aligned} \tag{5.28}$$

Turning to the second term in the right-hand side of (5.27), we notice that the estimate on the terms $R_{K,\sigma}$ in [11, Proof of Theorem 3.4] show that

$$\left|\frac{\widehat{\delta}_\sigma\phi}{d_\sigma}-\nabla\phi\cdot n_\sigma\right| \le Ch\frac{\sqrt{|\sigma|}}{\sqrt{d_\sigma}}\|\mathcal{H}\phi\|_{L^2(\cup_{L\in\mathcal{M}_\sigma}L)^{d\times d}}.$$

Hence, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\sum_{\sigma\in\mathcal{F}}\delta_\sigma v_{\mathcal{D}}\int_\sigma\left(\frac{\widehat{\delta}_\sigma\phi}{d_\sigma}-\nabla\phi\cdot n_\sigma\right)\mathrm{d}s(\boldsymbol{x})\right| \le Ch\|\mathcal{H}\phi\|\left(\sum_{\sigma\in\mathcal{F}}\frac{|\sigma|}{d_\sigma}(\delta_\sigma v_{\mathcal{D}})^2\right)^{1/2} \\ = Ch\|\phi\|_{H^2(\Omega)}\|v_{\mathcal{D}}\|_{\mathcal{D}} \le Ch\mathrm{diam}(\Omega)\|\phi\|_{H^2(\Omega)}\|\Delta_{\mathcal{D}}v_{\mathcal{D}}\|,\end{aligned} \tag{5.29}$$

where we have used (5.11) in the last line. Plugging (5.28) and (5.29) into (5.27), we obtain

$$\left|\int_\Omega \Delta\phi\Pi_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x} - \int_\Omega\phi\Delta_{\mathcal{D}}v_{\mathcal{D}}\,\mathrm{d}\boldsymbol{x}\right| \le Ch\|\phi\|_{H^2(\Omega)}\|\Delta_{\mathcal{D}}v_{\mathcal{D}}\|,$$

and the estimate on $W_{\mathcal{D}}(\xi)$ then follows from (5.24), recalling that $\phi = \mathrm{tr}(\xi)$. □

Remark 5.6. *The same analysis also probably applies to the second method presented in* [12, Section 5]*, which is applicable on general polygonal meshes.*

6. Numerical results

In this section, we present the results of some numerical experiments for the gradient recovery (GR) method and finite volume (FV) method presented in Sections 4 and 5. All these tests are conducted on the biharmonic problem $\Delta^2\overline{u} = f$ on $\Omega = (0,1)^2$, with clamped boundary conditions and for various exact solutions $\overline{u}$.

6.1. Numerical results for Gradient Recovery method.

Three examples are presented to illustrate the theoretical estimates of Theorem 3.6 on the Hessian discretisation described in Section 4.2. The considered FE space V_h is therefore the conforming $\mathbb{P}_1$ space, and the implementation was done following the ideas in [20]. The following relative errors, and related orders of convergence, in $L^2(\Omega)$, $H^1(\Omega)$

and $H^2(\Omega)$ norms are presented:

$$\mathsf{err}_{\mathcal{D}}(\overline{u}) := \frac{\|\Pi_{\mathcal{D}} u_{\mathcal{D}} - \overline{u}\|}{\|\overline{u}\|}, \quad \mathsf{err}(\nabla\overline{u}) := \frac{\|\nabla u_{\mathcal{D}} - \nabla\overline{u}\|}{\|\nabla\overline{u}\|}$$
$$\mathsf{err}_{\mathcal{D}}(\nabla\overline{u}) := \frac{\|\nabla_{\mathcal{D}} u_{\mathcal{D}} - \nabla\overline{u}\|}{\|\nabla\overline{u}\|} = \frac{\|Q_h \nabla u_{\mathcal{D}} - \nabla\overline{u}\|}{\|\nabla\overline{u}\|},$$
$$\mathsf{err}_{\mathcal{D}}(\mathcal{H}\overline{u}) := \frac{\|\mathcal{H}_{\mathcal{D}}^B u_{\mathcal{D}} - \mathcal{H}\overline{u}\|}{\|\mathcal{H}\overline{u}\|} = \frac{\|\nabla(Q_h \nabla u_{\mathcal{D}}) - \mathcal{H}\overline{u}\|}{\|\mathcal{H}\overline{u}\|},$$

where $u_{\mathcal{D}}$ is the solution to the Hessian scheme (3.1).
We provide in Table 1 the mesh data: mesh sizes h, numbers of unknowns (that is, the number of internal vertices) **nu**, and numbers of non-zero terms **nnz** in the square matrix of the system.

TABLE 1. (GR) Mesh size, number of unknowns and number of non-zero terms in the square matrix

h	**nu**	**nnz**
0.176777	9	79
0.088388	49	1203
0.044194	225	7011
0.022097	961	32835
0.011049	3969	141315
0.005524	16129	585603

6.1.1. *Example 1.* The exact solution is chosen to be $\overline{u}(x,y) = x^2(x-1)^2y^2(y-1)^2$. To assess the effect of the stabilisation function $\mathfrak{S}_h$ on the results, we multiply it by a factor r that takes the values 0.1, 1, 10, and 100.
The errors and orders of convergence for the numerical approximation to $\overline{u}$ are shown in Tables 2–5. It can be seen that the rate of convergence is quadratic in L^2-norm and linear in H^1-norm (see $\mathsf{err}(\nabla\overline{u})$). However, using gradient recovery operator, a quadratic order of convergence in H^1 norm is recovered (see $\mathsf{err}_{\mathcal{D}}(\nabla\overline{u})$). The rate of convergence in energy norm is linear (see $\mathsf{err}_{\mathcal{D}}(\mathcal{H}\overline{u})$), as expected by plugging the estimates of Theorem 4.3 into Theorem 3.6. We also notice a very small effect of r on the relative errors and rates.

TABLE 2. (GR) Convergence results for the relative errors, Example 1, $r = 0.1$

nu	$\mathsf{err}_{\mathcal{D}}(\overline{u})$	Order	$\mathsf{err}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\mathcal{H}\overline{u})$	Order
9	9.274702	-	31.591906	-	0.568338	-	0.595635	-
49	0.220095	5.3971	0.682922	5.5317	0.164105	1.7921	0.266927	1.1580
225	0.066997	1.7160	0.201282	1.7625	0.049395	1.7322	0.128410	1.0557
961	0.019135	1.8079	0.088805	1.1805	0.013697	1.8505	0.062164	1.0466
3969	0.005133	1.8983	0.040845	1.1205	0.003623	1.9185	0.030457	1.0293
16129	0.001331	1.9474	0.019422	1.0724	0.000933	1.9568	0.015059	1.0161

6.1.2. *Example 2.* We consider here the transcendental exact solution $\overline{u} = x^2(x-1)^2y^2(y-1)^2(\cos(2\pi x) + \sin(2\pi y))$, and $r = 0.1, 1$ and 10. Tables 6–8 presents the numerical results. The same comments as in Example 1 can be made about the rates of convergence. Past the coarsest meshes, we also notice as in Example 1 that r only has a small impact on the relative errors.

TABLE 7. (GR) Convergence results for the relative errors, Example 2, $r = 1$

nu	$\mathsf{err}_{\mathcal{D}}(\overline{u})$	Order	$\mathsf{err}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\mathcal{H}\overline{u})$	Order
9	10.222667	-	19.376883	-	1.058048	-	1.333720	-
49	0.475973	4.4247	1.467316	3.7231	0.229176	2.2069	0.473233	1.4948
225	0.074399	2.6775	0.313397	2.2271	0.050755	2.1748	0.170477	1.4730
961	0.017711	2.0706	0.112806	1.4742	0.013591	1.9009	0.079552	1.0996
3969	0.004547	1.9615	0.052162	1.1128	0.003640	1.9006	0.039162	1.0224
16129	0.001164	1.9657	0.025515	1.0317	0.000949	1.9393	0.019456	1.0092

TABLE 8. (GR) Convergence results for the relative errors, Example 2, $r = 10$

nu	$\mathsf{err}_{\mathcal{D}}(\overline{u})$	Order	$\mathsf{err}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\mathcal{H}\overline{u})$	Order
9	1.413122	-	2.541143	-	0.845365	-	0.894504	-
49	0.313425	2.1727	0.878752	1.5319	0.225247	1.9081	0.396725	1.1729
225	0.066842	2.2293	0.262354	1.7439	0.051757	2.1217	0.165546	1.2609
961	0.016897	1.9840	0.109794	1.2567	0.013783	1.9089	0.079311	1.0616
3969	0.004376	1.9492	0.052012	1.0779	0.003675	1.9072	0.039149	1.0185
16129	0.001123	1.9621	0.025506	1.0280	0.000956	1.9425	0.019455	1.0088

TABLE 9. (GR) Convergence results for the relative errors, Example 3, $r = 0.1$

nu	$\mathsf{err}_{\mathcal{D}}(\overline{u})$	Order	$\mathsf{err}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\mathcal{H}\overline{u})$	Order
9	81.804173	-	164.358300	-	1.068682	-	1.155266	-
49	0.677743	6.9153	2.358209	6.1230	0.232374	2.2013	0.517095	1.1597
225	0.093340	2.8602	0.447143	2.3989	0.048701	2.2544	0.207642	1.3163
961	0.017130	2.4459	0.125296	1.8354	0.010361	2.2328	0.084719	1.2933
3969	0.003975	2.1074	0.053941	1.2159	0.002643	1.9711	0.041197	1.0401
16129	0.000982	2.0167	0.026457	1.0278	0.000692	1.9341	0.020529	1.0049

TABLE 10. (GR) Convergence results for the relative errors, Example 3, $r = 1$

nu	$\mathsf{err}_{\mathcal{D}}(\overline{u})$	Order	$\mathsf{err}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\nabla\overline{u})$	Order	$\mathsf{err}_{\mathcal{D}}(\mathcal{H}\overline{u})$	Order
9	8.708395	-	16.990965	-	0.950590	-	0.990455	-
49	0.516904	4.0744	1.490046	3.5113	0.224877	2.0797	0.492555	1.0078
225	0.089332	2.5326	0.414243	1.8468	0.048056	2.2263	0.203301	1.2767
961	0.016920	2.4005	0.122315	1.7599	0.010349	2.2153	0.084441	1.2676
3969	0.003953	2.0975	0.053813	1.1846	0.002646	1.9678	0.041186	1.0358
16129	0.000978	2.0153	0.026452	1.0246	0.000693	1.9337	0.020528	1.0045

[15]. To ensure the correct orthogonality property (see Definition 5.1), the point $\boldsymbol{x}_K \in K$ is chosen as the circumcenter of K if K is a triangle, or the center of mass of K if K is a rectangle. As a result, for triangular meshes, the L^2 error, $\mathsf{err}_{\mathcal{D}}(\overline{u})$, is calculated using a skewed midpoint rule, where we consider the circumcenter of each cell instead of its center of mass. We denote the relative H^2 error by

$$\mathsf{err}_{\mathcal{D}}(\Delta\overline{u}) := \frac{\|\Delta_{\mathcal{D}}\overline{u}_{\mathcal{D}} - \Delta\overline{u}\|}{\|\Delta\overline{u}\|}.$$

the function values and first partial derivatives at the three vertices of the original triangle in addition to the normal derivative at the midpoints of the sides of the original triangle.

7.2. **An example of non-conforming method: the Adini rectangle.** Assume that Ω can be covered by mesh $\mathcal{M}$ made up of rectangles (we restrict the presentation to $d = 2$ for simplicity). The element K consists of a rectangle with vertices $\{a_i, 1 \le i \le 4\}$; the space $\mathbb{P}_K$ is given by $\mathbb{P}_K = \mathbb{P}_3 \oplus \{x_1 x_2^3\} \oplus \{x_1^3 x_2\}$, by which we mean polynomials of degree ≤ 4 whose only fourth-degree terms are those involving $x_1 x_2^3$ and $x_1^3 x_2$. Thus $\mathbb{P}_3 \subset \mathbb{P}_K$. The set of degrees of freedom in each cell is

$$\Sigma_K = \left\{ p(a_i), \frac{\partial p}{\partial x_1}(a_i), \frac{\partial p}{\partial x_2}(a_i); 1 \le i \le 4\,,\ p \in \mathbb{P}_K \right\}.$$

The global approximation space is then given by

$$\begin{aligned} V_h =: \{ & v_h \in L^2(\Omega);\ v_h|_K \in \mathbb{P}_K\, \forall\, K \in \mathcal{M}, v_h \text{ and } \nabla v_h \text{ are continuous at} \\ & \text{the vertices of elements in } \mathcal{M}, v_h \text{ and } \nabla v_h \text{ vanish at vertices on } \partial\Omega\}. \end{aligned}$$

Note that $V_h \subset H_0^1(\Omega) \cap C^0(\overline{\Omega})$.

Definition 7.1 (Hessian discretisation for the Adini rectangle)**.** *Each $v_{\mathcal{D}} \in X_{\mathcal{D},0}$ is a vector of three values at each vertex of the mesh (with zero values at boundary vertices), corresponding to function and gradient values, $\Pi_{\mathcal{D}} v_{\mathcal{D}}$ is the function such that $(\Pi_{\mathcal{D}} v_{\mathcal{D}})_{|K} \in \mathbb{P}_K$ and its gradient takes the values at the vertices dictated by $v_{\mathcal{D}}$, $\nabla_{\mathcal{D}} v_{\mathcal{D}} = \nabla(\Pi_{\mathcal{D}} v_{\mathcal{D}})$ and $\mathcal{H}_{\mathcal{D}}^B v_{\mathcal{D}} = \mathcal{H}_{\mathcal{M}}^B(\Pi_{\mathcal{D}} v_{\mathcal{D}})$ is the broken $\mathcal{H}^B$ ($\mathcal{H}_{\mathcal{D}}$ is the broken $\mathcal{H}$).*

We assume that the mesh is regular, that is, (4.1) holds with η not depending on the mesh.

Theorem 7.2. *Let $\mathcal{D}$ be a B–Hessian discretisation in the sense of Definition 7.1 with B satisfying the coercive property. Then, there exists a constant C, not depending on $\mathcal{D}$, such that*

- $C_{\mathcal{D}}^B \le C$,
- $\forall \varphi \in H^3(\Omega) \cap H_0^2(\Omega)$, $S_{\mathcal{D}}^B(\varphi) \le Ch\|\varphi\|_{H^3(\Omega)}$,
- $\forall \xi \in H^2(\Omega)^{d\times d}$, $W_{\mathcal{D}}^B(\xi) \le Ch\|\xi\|_{H^2(\Omega)^{d\times d}}$.

The properties of Hessian discretisations built on the Adini rectangle follow from this theorem and Remark 3.4.

Corollary 7.3. *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a sequence of B–Hessian discretisations built on the Adini rectangle, such that B is coercive and the underlying sequence of meshes are regular and have a size that goes to 0 as $m \to \infty$. Then the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is coercive, consistent and limit-conforming.*

Proof of Theorem 7.2.
In this proof, $C > 0$ denotes a generic constant that can change from one line to the other but depends only on Ω, d, B and η.

• Coercivity: Since $V_h \subset H_0^1(\Omega)$, for $v \in X_{\mathcal{D},0}$, the Poincaré inequality yields $\|\Pi_{\mathcal{D}} v\| \le \operatorname{diam}(\Omega)\|\nabla_{\mathcal{D}} v\|$, which gives us part of the estimate on $C_{\mathcal{D}}^B$. Define the *broken Sobolev* space

$$H^1(\mathcal{M}) = \left\{ v \in L^2(\Omega)\,;\, \forall K \in \mathcal{M}, v_{|K} \in H^1(K) \right\}$$

and endow it with the dG norm

$$\|w\|_{dG}^2 := \|\nabla_{\mathcal{M}} w\|^2 + \sum_{\sigma\in\mathcal{F}} \frac{1}{h_\sigma} \|[\![w]\!]\|_{L^2(\sigma)}^2, \tag{7.1}$$

where

$$h_\sigma = \begin{cases} \min(h_K, h_L) & \text{if } \sigma \in \mathcal{F}_{\rm int},\ \mathcal{M}_\sigma = \{K, L\} \\ h_K & \text{if } \sigma \in \mathcal{F}_{\rm ext},\ \mathcal{M}_\sigma = K, \end{cases}$$

and the jump of w is

$$[\![w]\!] = \begin{cases} w_{|K} - w_{|L} & \text{if } \sigma \in \mathcal{F}_{\rm int},\ \mathcal{M}_\sigma = \{K, L\} \\ w_{|K} & \text{if } \sigma \in \mathcal{F}_{\rm ext},\ \mathcal{M}_\sigma = K. \end{cases}$$

If $[\![w]\!] = 0$ at the vertices of σ then, by the Poincaré inequality in $H_0^1(\sigma)$ (Lemma A.1),

$$\|[\![w]\!]\|_{L^2(\sigma)} \le C h_\sigma \|\nabla_{\mathcal{M}} [\![w]\!]\|_{L^2(\sigma)^d}. \tag{7.2}$$

If $\sigma \in \mathcal{F}_{\rm int}$ with $\mathcal{M}_\sigma = \{K, L\}$ then $[\![w]\!] = 0$ at the vertices of σ, and (7.2) combined with the trace inequality [7, Lemma 1.46] therefore give

$$\begin{aligned} \|[\![w]\!]\|_{L^2(\sigma)} &\le C h_\sigma (\|\nabla_{\mathcal{M}} w_{|K}\|_{L^2(\sigma)^d} + \|\nabla_{\mathcal{M}} w_{|L}\|_{L^2(\sigma)^d}) \\ &\le C_{\rm tr} h_\sigma (h_K^{-1/2} \|\nabla_{\mathcal{M}} w\|_{L^2(K)^d} + h_L^{-1/2} \|\nabla_{\mathcal{M}} w\|_{L^2(L)^d}), \end{aligned} \tag{7.3}$$

where $C_{\rm tr}$ depends only on d and the mesh regularity parameter η. Take $v \in X_{\mathcal{D},0}$. Since $\nabla_{\mathcal{D}} v$ is continuous at the vertices of elements in $\mathcal{M}$ and $\nabla_{\mathcal{D}} v$ vanish at vertices along $\partial\Omega$, choosing $w = \nabla_{\mathcal{D}} v$ in (7.2) and (7.3) yields

$$\|[\![\nabla_{\mathcal{D}} v]\!]\|_{L^2(\sigma)^d} \le C_{\rm tr} h_\sigma \Big(h_K^{-1/2} \|\nabla_{\mathcal{M}}(\nabla_{\mathcal{D}} v)\|_{L^2(K)^{d\times d}} + h_L^{-1/2} \|\nabla_{\mathcal{M}}(\nabla_{\mathcal{D}} v)\|_{L^2(L)^{d\times d}} \Big).$$

Recalling the definition (7.1) of the dG norm, the above inequality and the coercivity property of B yield

$$\begin{aligned} \|\nabla_{\mathcal{D}} v\|_{dG}^2 \le{}& \|\nabla_{\mathcal{M}}(\nabla_{\mathcal{D}} v)\|^2 \\ &+ 2C_{\rm tr} \sum_{\sigma\in\mathcal{F}} h_\sigma \Big(h_K^{-1} \|\nabla_{\mathcal{M}}(\nabla_{\mathcal{D}} v)\|_{L^2(K)^{d\times d}}^2 + h_L^{-1} \|\nabla_{\mathcal{M}}(\nabla_{\mathcal{D}} v)\|_{L^2(L)^{d\times d}}^2 \Big) \\ \le{}& \|\nabla_{\mathcal{M}}(\nabla_{\mathcal{D}} v)\|^2 + C \sum_{K\in\mathcal{M}} \|\nabla_{\mathcal{M}}(\nabla_{\mathcal{D}} v)\|_{L^2(K)^{d\times d}}^2 \\ \le{}& C\|\mathcal{H}_{\mathcal{M}}(\Pi_{\mathcal{D}} v)\|^2 \le C\varrho^{-2} \|\mathcal{H}_{\mathcal{M}}^B(\Pi_{\mathcal{D}} v)\|^2 = C\varrho^{-2}\|\mathcal{H}_{\mathcal{D}}^B v\|^2. \end{aligned}$$

Using the fact that $\|w\| \le C\|w\|_{dG}$ whenever w is a broken polynomial on $\mathcal{M}$ (see [7, Theorem 5.3]), we infer that $\|\nabla_{\mathcal{D}} v\| \le C\varrho^{-1}\|\mathcal{H}_{\mathcal{D}}^B v\|$, which concludes the estimate on $C_{\mathcal{D}}^B$.

• Consistency: Consistency follows from the affine property of the family of Adini rectangles. Using [6, Theorem 3.1.5, Chapter 3], for $\varphi \in H^3(\Omega) \cap H_0^2(\Omega)$, we obtain

$$\inf_{w\in X_{\mathcal{D},0}} \|\mathcal{H}_{\mathcal{D}}^B w - \mathcal{H}^B \varphi\| \le Ch|\phi|_{3,\Omega}\,, \qquad \inf_{w\in X_{\mathcal{D},0}} \|\nabla_{\mathcal{D}} w - \nabla\varphi\| \le Ch^2|\phi|_{3,\Omega}$$
$$\text{and} \qquad \inf_{w\in X_{\mathcal{D},0}} \|\Pi_{\mathcal{D}} w - \varphi\| \le Ch^3|\phi|_{3,\Omega},$$

which implies $S_{\mathcal{D}}^B(\varphi) \le Ch|\phi|_{3,\Omega}$.

• LIMIT-CONFORMITY: for $\xi \in H^2(\Omega)^{d\times d}$ and $v_{\mathcal{D}} \in X_{\mathcal{D},0}$, cellwise integration-by-parts (see Lemma A.2) yields

$$\begin{aligned}\int_\Omega (\mathcal{H} : B^\tau B\xi)\Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} &= \sum_{K\in\mathcal{M}} \int_K (\mathcal{H} : A\xi)\Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} \\ &= \int_\Omega A\xi : \mathcal{H}_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} - \sum_{K\in\mathcal{M}} \int_{\partial K} (A\xi n_K)\cdot \nabla_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}s(\boldsymbol{x}) \\ &\quad + \sum_{K\in\mathcal{M}} \int_{\partial K} (\mathrm{div}(A\xi)\cdot n_K)\Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}s(\boldsymbol{x}).\end{aligned}$$

For $K\in\mathcal{M}$ and $\sigma\in\mathcal{F}_K$, let $n_{K,\sigma}$ be the unit vector normal to σ outward to K. For all $\sigma\in\mathcal{F}$, we choose an orientation (that is, a cell K such that $\sigma\in\mathcal{F}_K$) and we set $n_\sigma = n_{K,\sigma}$. We then set $[\![w]\!] = w_{|K} - w_{|L}$ if $\sigma\in\mathcal{F}_{\rm int}$ with $\mathcal{M}_\sigma = \{K,L\}$, and $[\![w]\!] = w_{|K}$ if $\sigma\in\mathcal{F}_{\rm ext}$ with $\mathcal{M}_\sigma = K$. Then

$$\begin{aligned}&\int_\Omega (\mathcal{H} : A\xi)\Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} - \int_\Omega A\xi : \mathcal{H}_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} \\ &= -\sum_{\sigma\in\mathcal{F}} \int_\sigma (A\xi n_\sigma)\cdot [\![\nabla_{\mathcal{D}} v_{\mathcal{D}}]\!] \,\mathrm{d}s(\boldsymbol{x}) + \sum_{\sigma\in\mathcal{F}} \int_\sigma (\mathrm{div}(A\xi)\cdot n_\sigma)[\![\Pi_{\mathcal{D}} v_{\mathcal{D}}]\!] \,\mathrm{d}s(\boldsymbol{x}).\end{aligned} \tag{7.4}$$

Since $\Pi_{\mathcal{D}} v_{\mathcal{D}} \in H^1_0(\Omega)\cap C(\overline{\Omega})$, $[\![\Pi_{\mathcal{D}} v_{\mathcal{D}}]\!] = 0$. Let Λ_K denote the Q_1 interpolation operator associated with the values at the four vertices of K, and Λ_h be the patched interpolator such that $(\Lambda_h)_{|K} = \Lambda_K$ for all K. $\Lambda_h(\nabla_{\mathcal{D}} v_{\mathcal{D}})$ takes the values of $\nabla_{\mathcal{D}} v_{\mathcal{D}}$ at the vertices, so it is continuous at internal vertices and vanishes at the boundary vertices. Hence, for any $\sigma\in\mathcal{F}$, $[\![\Lambda_h(\nabla_{\mathcal{D}} v_{\mathcal{D}})]\!]$ vanishes on σ since it is linear on this edge and vanishes at its vertices. As a consequence,

$$\begin{aligned}&\int_\Omega (\mathcal{H} : A\xi)\Pi_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} - \int_\Omega A\xi : \mathcal{H}_{\mathcal{D}} v_{\mathcal{D}} \,\mathrm{d}\boldsymbol{x} \\ &\qquad = -\sum_{\sigma\in\mathcal{F}} \int_\sigma (A\xi n_\sigma)\cdot [\![\nabla_{\mathcal{D}} v_{\mathcal{D}} - \Lambda_h(\nabla_{\mathcal{D}} v_{\mathcal{D}})]\!] \,\mathrm{d}s(\boldsymbol{x}) \\ &\qquad = -\sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{F}_K} \int_\sigma A\xi n_{K,\sigma}\cdot \Big(\nabla_{\mathcal{D}} v_{\mathcal{D}} - \Lambda_K(\nabla_{\mathcal{D}} v_{\mathcal{D}})\Big) \,\mathrm{d}s(\boldsymbol{x}).\end{aligned} \tag{7.5}$$

Setting $\varphi = A\xi n_{K,\sigma}$ and $w = \nabla_{\mathcal{D}} v_{\mathcal{D}}$, a change of variables yields

$$\int_{\sigma\in\mathcal{F}_K} \varphi\cdot\big(w - \Lambda_K(w)\big)\,\mathrm{d}s(\boldsymbol{x}) = |\sigma| \int_{\widehat{\sigma}\in\mathcal{F}_{\widehat{K}}} \widehat{\varphi}\cdot\big(\widehat{w} - \Lambda_{\widehat{K}}(\widehat{w})\big)\,\mathrm{d}s(\boldsymbol{x}), \tag{7.6}$$

where $\widehat{K}$ is the reference finite element. Let $\mathcal{F}_K = \{\sigma_1^{'}, \sigma_2^{'}, \sigma_1^{''}, \sigma_2^{''}\}$ such that $|\sigma_1^{'}| = |\sigma_1^{''}| = h_1$ and $|\sigma_2^{'}| = |\sigma_2^{''}| = h_2$. Let us consider

$$\delta_{1,K}(\phi, v) = \int_{\sigma_1^{'}} \phi\big(v - \Lambda_K(v)\big)\,\mathrm{d}s(\boldsymbol{x}) - \int_{\sigma_1^{''}} \phi\big(v - \Lambda_K(v)\big)\,\mathrm{d}s(\boldsymbol{x}), \tag{7.7}$$

for $\phi\in H^1(K)$ and $v\in\partial_1\mathbb{P}_K$. The steps in [6, Theorem 6.2.3] show that $\delta_{1,K}(\phi,v) \le Ch|\phi|_{1,K}|v|_{1,K}$. For the sake of completeness, let us briefly recall the argument. Using changes of variables, $\delta_{1,K}(\phi,v) = h_1\delta_{1,\widehat{K}}(\widehat{\phi},\widehat{v})$. Since $\mathbb{P}_0\subset Q_1$, which is preserved by Λ_K, for all $\widehat{v}\in\mathbb{P}_0$ and $\widehat{\phi}\in H^1(\widehat{K})$ we have $\delta_{1,\widehat{K}}(\widehat{\phi},\widehat{v}) = 0$ (first polynomial invariance). Let us now prove that the same relation holds if

$\widehat{\phi} \in \mathbb{P}_0$ and $\widehat{v} \in \partial_1 P_{\widehat{K}}$. Since $\widehat{\phi} \in \mathbb{P}_0$, its value on $\widehat{K}$ is a constant, say, equal to a_0. Since $\widehat{v} \in \partial_1 P_{\widehat{K}}$ we have

$$\widehat{v} = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1^2 + b_4 x_1 x_2 + b_5 x_2^2 + b_6 x_1^2 x_2 + b_7 x_2^3.$$

Taking the values at the four vertices, we get

$$\Lambda_{\widehat{K}} \widehat{v} = b_0 + (b_1 + b_3) x_1 + (b_2 + b_5 + b_7) x_2 + (b_4 + b_6) x_1 x_2.$$

Assuming without loss of generality that $\sigma_1^{'}$ is the line $x_1 = 1$ and $\sigma_1^{''}$ is the line $x_1 = 0$, we infer

$$(\widehat{v} - \Lambda_{\widehat{K}} \widehat{v})_{|x_1=0} = -(b_5 + b_7) x_2 + b_5 x_2^2 + b_7 x_2^3,$$
$$(\widehat{v} - \Lambda_{\widehat{K}} \widehat{v})_{|x_1=1} = -(b_5 + b_7) x_2 + b_5 x_2^2 + b_7 x_2^3.$$

The relation $\delta_{1,\widehat{K}}(\widehat{\phi}, \widehat{v}) = 0$ (second polynomial invariance) then follows from

$$\begin{aligned}\int_{\sigma_1^{'}} \widehat{\phi}(\widehat{v} - \Lambda_{\widehat{K}} \widehat{v})\, \mathrm{d}s(\boldsymbol{x}) &= \int_0^1 a_0(-(b_5 + b_7) x_2 + b_5 x_2^2 + b_7 x_2^3)\, \mathrm{d}x_2 \\ &= \int_{\sigma_1^{''}} \widehat{\phi}(\widehat{v} - \Lambda_{\widehat{K}} \widehat{v})\, \mathrm{d}s(\boldsymbol{x}).\end{aligned}$$

The bilinear form $\delta_{1,\widehat{K}}(\widehat{\phi}, \widehat{v})$ is continuous over the space $H^1(\widehat{K}) \times \partial_1 P_{\widehat{K}}$ by the trace theorem. Using the bilinear lemma [6, Theorem 4.2.5], we deduce from the two polynomial invariances the existence of a constant C such that $|\delta_{1,\widehat{K}}(\widehat{\phi}, \widehat{v})| \le C|\widehat{\phi}|_{1,\widehat{K}} |\widehat{v}|_{1,\widehat{K}}$ for all $\widehat{\phi} \in H^1(\widehat{K})$, $\widehat{v} \in \partial_1 P_{\widehat{K}}$. A direct change of variables shows that

$$|\widehat{\phi}|_{1,\widehat{K}} \le C|\phi|_{1,K} \quad \text{and} \quad |\widehat{v}|_{1,\widehat{K}} \le C|v|_{1,K}.$$

Since $\delta_{1,K}(\phi, v) = h_1 \delta_{1,\widehat{K}}(\widehat{\phi}, \widehat{v})$, we infer $\delta_{1,K}(\phi, v) \le Ch|\phi|_{1,K}|v|_{1,K}$. Similarly, $\delta_{2,K}(\phi, v) \le Ch|\phi|_{1,K}|v|_{1,K}$ (considering integrals over $\sigma_2^{'}$ and $\sigma_2^{''}$). Hence, from (7.5), (7.6) and (7.7),

$$\left| \int_\Omega (\mathcal{H} : A\xi) \Pi_{\mathcal{D}} v_{\mathcal{D}}\, \mathrm{d}\boldsymbol{x} - \int_\Omega A\xi : \mathcal{H}_{\mathcal{D}} v_{\mathcal{D}}\, \mathrm{d}\boldsymbol{x} \right| \le C\|\xi\|_{H^2(\Omega)^{d\times d}} h \|\mathcal{H}_{\mathcal{D}}^B v_{\mathcal{D}}\|.$$

The proof of the estimate on $W_{\mathcal{D}}^B(\xi)$ is complete. □

Appendix A. Technical results

Lemma A.1 (Poincaré inequality along an edge)**.** *Let σ be an edge of a polygonal cell, $w \in H^1(\sigma)$ and assume that w vanish at a point on the edge $\sigma \in \mathcal{F}$. Then there exists $C > 0$ such that*

$$\|w\|_{L^2(\sigma)} \le h_\sigma \|\partial w\|_{L^2(\sigma)},$$

where ∂ denotes the derivative along the edge and h_σ is the length of the edge.

Proof. Let m denote the point on the edge σ which satisfies $w(m) = 0$. For $m < x$, we get

$$w(x) = w(m) + \int_m^x \partial w(y)\, \mathrm{d}y = \int_m^x \partial w(y)\, \mathrm{d}y.$$

A use of Cauchy-Schwarz inequality yields

$$|w(x)| \le |x - m|^{1/2} \left(\int_m^x |\nabla w|^2\, \mathrm{d}y \right)^{1/2} \le \sqrt{h_\sigma} \left(\int_\sigma |\partial w|^2\, \mathrm{d}y \right)^{1/2}.$$

Squaring this yields $|w(x)|^2 \leq h_\sigma \int_\sigma |\partial w|^2 \,\mathrm{d}y$ and integrating over the edge concludes the proof. $\square$

Lemma A.2 (Integration by parts)**.** *Let P be a fourth order tensor. For $\xi \in H^2(\Omega)^{d\times d}$ and $\phi \in H^1(\Omega)$, we have*

$$\int_\Omega (\mathcal{H} : P\xi)\phi = -\int_\Omega \nabla\phi \cdot \mathrm{div}(P\xi) + \int_{\partial\Omega} \mathrm{div}(P\xi \cdot n)\phi.$$

For $\psi \in H^2(\Omega)$,

$$\int_\Omega P\xi : \mathcal{H}\psi = -\int_\Omega \nabla\psi \cdot \mathrm{div}(P\xi) + \int_{\partial\Omega} (\mathrm{div}(P\xi n)) \cdot \nabla\psi.$$

For $\zeta \in H^1(\Omega)^d$,

$$\int_\Omega P\xi : \nabla\zeta = -\int_\Omega \mathrm{div}(P\xi) \cdot \zeta + \int_{\partial\Omega} (\mathrm{div}(P\xi n)) \cdot \zeta.$$

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